2-NILPOTENT AND RIGID FINITE DIMENSIONAL ALGEBRAS

CLAUDE CIBILS

1. *Introduction*

Let k be a field and let Λ be a finite dimensional k-algebra. We suppose that Λ is Morita reduced and that the endomorphism ring of each simple Λ-module is k. This is equivalent to $\Lambda/r = k \times \ldots \times k$, where r is the Jacobson radical of Λ.

Let Q be the Gabriel quiver of Λ which is defined as follows: the set Q_0 of vertices is in one-to-one correspondence with the set of isomorphism classes of simple Λ-modules. If a simple Λ-module is denoted by S, we denote by s the corresponding vertex. The number of arrows from s to t is the k-dimension of $\mathrm{Ext}^1_\Lambda(S, T)$. The set Q_1 of arrows of the quiver is the disjoint union of the set B of loops and the set A of oriented edges.

Let kQ be the quiver algebra. By an observation of Gabriel [**4**; **5**, 4.3] we have an algebra surjection $kQ \to \Lambda$ whose kernel I is admissible, that is, one has $F^n \subset I \subset F^2$, where F is the two sided ideal of kQ generated by the arrows, and n is some positive integer.

In this way each finite dimensional k-algebra Λ such that $\Lambda/r = k \times \ldots \times k$ is isomorphic to kQ/I with I admissible. The radical r of kQ/I is F/I.

By definition the algebra Λ is 2-nilpotent if $r^2 = 0$. In the previous description this is equivalent to $I = F^2$.

Let $H^*(\Lambda, X)$ denote the Hochschild cohomology groups of Λ with coefficients in the Λ-bimodule X (see [**2**] for a definition).

In this note we give a description of the algebras $\Lambda = kQ/F^2$ such that $H^2(\Lambda, \Lambda) = 0$. Supposing Q is connected, the precise conditions are that Q does not contain loops nor unoriented triangles

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \end{tikzcd}$$

and Q is not the quiver $\bullet \rightleftarrows \bullet$.

Let us recall that for hereditary algebras, $H^2(kQ, X) = 0$ for all quivers Q without oriented cycles and all kQ-bimodules X, see [**6**, **3**].

If k is algebraically closed, the orbit of a k-algebra Λ of dimension n under the natural action of GL_n is open (in the scheme sense) if and only if $H^2(\Lambda, \Lambda) = 0$ and the algebraic group $\mathrm{Aut}\,\Lambda$ of automorphisms of Λ is smooth. When the characteristic is zero, the orbit of Λ is open if and only if $H^2(\Lambda, \Lambda) = 0$, because then the algebraic groups are smooth (see [**6**]).

In [**7**] Gerstenhaber shows that if $H^2(\Lambda, \Lambda) = 0$, then Λ is rigid, that is, each

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one-parameter family of deformations of Λ is trivial. We shall prove directly that the given conditions on Q imply that $\Lambda = kQ/F^2$ is rigid. I thank the referee for suggesting this proof, which is shorter than the original one.

The converse of the result is proved using the fact that a connecting homomorphism in a long exact sequence of cohomology groups is a cup product with a fixed element.

In [3] we gave sufficient conditions on the quiver Q and the ideal I of a finite dimensional algebra Λ for $H^2(\Lambda, \Lambda)$ to be zero. However these conditions are not necessary as this note shows; see the remark at the end.

Finally we remark that it is not hard to show that

$$\dim_k H^1(\Lambda, \Lambda) = \begin{cases} \sum\limits_{s,t \in Q_0} |tQ_1 s|^2 - |Q_0| + 1 & \text{if char k} \neq 2, \\ \sum\limits_{s,t \in Q_0} |tQ_1 s|^2 - |Q_0| + 1 + |B| & \text{if char} \, k = 2. \end{cases}$$

One uses the fact that $H^1(\Lambda, \Lambda)$ is the vector space of derivations of Λ modulo the interior derivations.

2. *The theorem*

Throughout this note k is an arbitrary field, Q is a finite connected quiver and F is the two sided ideal of kQ generated by the arrows of Q.

PROPOSITION. *Let $\Lambda = kQ/F^2$. If $H^2(\Lambda, \Lambda) = 0$ then Q does not contain loops nor subquivers of the type*

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \end{tikzcd}$$

called 'unoriented triangles', and Q is not the quiver $\bullet \rightleftarrows \bullet$

Proof. Let $K = k((x))$ be the quotient power series field of $k[[x]]$, the power series ring in one variable.

If Q contains a loop b, consider the two sided ideal I of KQ generated by $b^2 - xb$ and by all the paths of length two except b^2. The algebra $\Lambda_x = KQ/I$ is a non-trivial one-parameter family of deformations of $\Lambda = kQ/F^2$, as defined in [7]. In fact Λ_x is a K-algebra isomorphic to KQ'/F^2, where Q' is obtained from Q by deleting the loop b and taking the disjoint union with a vertex. This algebra is clearly not isomorphic to KQ/F^2, and so $H^2(\Lambda, \Lambda) \neq 0$ using the result of [7] quoted in the introduction.

If Q contains an unoriented triangle

$$\begin{tikzcd} & \bullet \arrow[dr, "b"] & \\ \bullet \arrow[ur, "a"] \arrow[rr, "c"'] & & \bullet \end{tikzcd}$$

let I be generated by $ba - xc$ and by all the paths of length two except ba. Again $\Lambda_x = KQ/I$ is a non-trivial one-parameter family of deformations of $\Lambda = kQ/F^2$. The K-algebra Λ_x is isomorphic to KQ'/R, where Q' is obtained from Q by deleting the arrow c and R is generated by all the paths of length two except ba.

If

$$Q = s \bullet \underset{b}{\overset{a}{\rightleftarrows}} \bullet \, t,$$

let I be the two sided ideal of KQ generated by $ba - xs$ and $ab - xt$, and let Λ_x be KQ/I. In fact Λ_x is isomorphic to the algebra of 2×2 matrices over K. Notice that we cannot extend this deformation for $Q = \bullet \rightleftarrows \bullet \longrightarrow \bullet$ and $\Lambda = kQ/F^2$. In fact we shall see that $H^2(\Lambda, \Lambda) = 0$ in this case.

The following lemma computes the dimension of $H^2(\Lambda, r)$ without the assumption that $H^2(\Lambda, \Lambda) = 0$ (but still for a 2-nilpotent algebra).

LEMMA 1. *Let $\Lambda = kQ/F^2$. Then*

$$\dim_k H^2(\Lambda, r) = \sum_{s, t \in Q_0} \sum_{u \in Q_0} |tQ_1 s|\,|tQ_1 u|\,|uQ_1 s|.$$

Proof. Let $\gamma \in Q_1$, and denote by s and t its source and end vertices. The one dimensional vector space $k\gamma$ is a simple Λ-bimodule in the following way: $e\gamma = \delta_{e,t}\gamma$ and $\gamma e = \delta_{e,s}\gamma$ for all e in the k-basis $Q_0 \cup Q_1$ of Λ.

We see that $r = \oplus_{\gamma \in Q_1} k\gamma$ is a direct decomposition of the semisimple Λ-bimodule r, and so

$$H^2(\Lambda, r) = \bigoplus_{\gamma \in Q_1} H^2(\Lambda, k\gamma).$$

Each of the simple Λ-bimodules $k\gamma$ is isomorphic to $\mathrm{Hom}_k(S, T)$, where S and T are the simple left Λ-modules corresponding to the vertices s and t. We obtain

$$H^2(\Lambda, r) = \bigoplus_{\gamma \in Q_1} H^2(\Lambda, \mathrm{Hom}_k(S, T)).$$

We then use the following result [2, p. 179]. If M and N are left Λ-modules, the Hochschild cohomology of the Λ-bimodule $\mathrm{Hom}_k(M, N)$ can be obtained using the functor Ext over Λ:

$$H^*(\Lambda, \mathrm{Hom}_k(M, N)) = \mathrm{Ext}^*_\Lambda(M, N).$$

Then $H^2(\Lambda, r) = \oplus_{\gamma \in Q_1} \mathrm{Ext}^2_\Lambda(S, T)$.

The dimension of this vector space has been computed by K. Bongartz in [1, p. 463] using a particular projective resolution. In our case the ideal of relations is F^2 and so we obtain

$$\begin{aligned} \dim_k \mathrm{Ext}^2_\Lambda(S, T) &= \dim_k t(F^2/F^3)s \\ &= \text{number of paths of length two from } s \text{ to } t \\ &= \sum_{u \in Q_0} |tQ_1 u|\,|uQ_1 s|. \end{aligned}$$

The number of $\mathrm{Ext}^2_\Lambda(S, T)$ direct summands in $H^2(\Lambda, r)$ is $|tQ_1 s|$, and the formula is proved.

Using this lemma we can prove the converse of the proposition and obtain the following.

THEOREM. *Let $\Lambda = kQ/F^2$, where Q is connected. Then $H^2(\Lambda, \Lambda) = 0$ if and only if Q does not contain loops, or unoriented triangles and Q is not the quiver $\bullet \rightleftarrows \bullet$.*

Proof. If Q is a quiver without loops and unoriented triangles, then Lemma 1 gives us that $H^2(\Lambda, r) = 0$. The long exact sequence of cohomology associated to

$$0 \longrightarrow r \longrightarrow \Lambda \longrightarrow \Lambda/r \longrightarrow 0$$

gives

$$0 \longrightarrow H^2(\Lambda, \Lambda) \longrightarrow H^2(\Lambda, \Lambda/r) \xrightarrow{\delta} H^3(\Lambda, r) \longrightarrow \cdots.$$

CLAIM. For the quivers as in the theorem, the connecting homomorphism δ is injective; thus $H^2(\Lambda,\Lambda)=0$. If Q is $\bullet \rightleftarrows \bullet$ then $1=\dim_k \operatorname{Ker}\delta=\dim_k H^2(\Lambda,\Lambda)$.

The rest of this note is devoted to the proof of the claim. We shall need a description of $H^2(\Lambda,\Lambda/r)$, $H^3(\Lambda,r)$ and of the homomorphism δ. The following lemma will give a description of δ using cup products. We choose to give it in a more general setting than just for 2-nilpotent algebras.

Let $\Lambda = kQ/I$, where I is an admissible ideal. Let $r=F/I$ be the radical of Λ, and let n be the nilpotence index of Λ, that is, $r^n=0$ and $r^{n-1}\neq 0$.

The product in Λ gives bimodule homomorphisms

$$r/r^2 \underset{\Lambda}{\otimes} r^{n-2}/r^{n-1} \longrightarrow r^{n-1}, \qquad r^{n-2}/r^{n-1} \underset{\Lambda}{\otimes} r/r^2 \longrightarrow r^{n-1},$$

which give cup products

$$H^1(\Lambda, r/r^2)\otimes H^i(\Lambda, r^{n-2}/r^{n-1}) \longrightarrow H^{i+1}(\Lambda, r^{n-1}),$$
$$H^i(\Lambda, r^{n-2}/r^{n-1})\otimes H^1(\Lambda, r/r^2) \longrightarrow H^{i+1}(\Lambda, r^{n-1}),$$

both denoted by $\cup$.

A set P of oriented paths of Q can be chosen such that P is a basis of Λ, and such that the subset P^i of oriented paths of length greater than or equal to i is a basis of r^i. The set Q_1 of arrows is always in P, and is a basis of r/r^2.

To each arrow a we can associate an element $f_a\in H^1(\Lambda, r/r^2)$ defined by $f_a(a)=a$ and $f_a(e)=0$ for all $e\in P$, $e\neq a$. It is easily checked that f_a is a non-interior derivation. Let us call $\mathscr{S}$ the element $\sum_{a\in Q_1} f_a \in H^1(\Lambda, r/r^2)$.

LEMMA 2. *Let $\Lambda = kQ/I$, where I is an admissible ideal of kQ. Consider the exact sequence*

$$0 \longrightarrow r^{n-1} \longrightarrow r^{n-2} \longrightarrow r^{n-2}/r^{n-1} \longrightarrow 0$$

and

$$\delta\colon H^{2i}(\Lambda, r^{n-2}/r^{n-1}) \longrightarrow H^{3i+1}(\Lambda, r^{n-1})$$

the connecting homomorphism.

Then $\delta(f)=\mathscr{S}\cup f+(-1)^{i+1} f\cup \mathscr{S}$.

Proof. Since P^{n-1} is a k-basis of r^{n-1} and P^{n-2} of r^{n-2}, it follows that $P_{n-2}=P^{n-2}\setminus P^{n-1}$ is a k-basis of r^{n-2}/r^{n-1}. We choose the k-section σ of the exact sequence corresponding to these bases, that is, $\sigma(\gamma)=\gamma$ if $\gamma\in P_{n-2}$.

In fact r^{n-2}/r^{n-1} is a semisimple Λ-bimodule and $\oplus_{\gamma\in P_{n-2}} k\gamma$ is its decomposition. Thus, let $f\colon \Lambda^{\otimes i}\to k\gamma$ be an i-cocycle. This means that for all $x_1,\dots,x_{i+1}$ in P we have

$$\sum_{j=1}^{i} (-1)^j f(x_1\otimes\dots\otimes x_j x_{j+1}\otimes\dots\otimes x_{i+1}) = -[x_1 f(x_2\otimes\dots\otimes x_i)+(-1)^{i+1} f(x_1\otimes\dots\otimes x_i)x_{i+1}]. \quad (*)$$

We know that $\delta(f)=d(\delta f)$, where d is the Hochschild coboundary from $\operatorname{Hom}_k(\Lambda^{\otimes i}, r^{n-2})$ to $\operatorname{Hom}_k(\Lambda^{\otimes i+1}, r^{n-2})$. In fact $d(\sigma f)\in \operatorname{Hom}_k(\Lambda^{\otimes i+1}, r^{n-1})$ and so we consider its cohomology class to obtain $\delta(f)$. Then we have

$$d(\gamma f)(x_1\otimes\dots\otimes x_{i+1}) = x_1(\sigma f)(x_2\otimes\dots\otimes x_i) + \sum_{j=1}^{i}(-1)^j(\sigma f)(x_1\otimes\dots\otimes x_j x_{j+1}\otimes\dots\otimes x_{i+1}) + (-1)^{i+1}(\sigma f)(x_1\otimes\dots\otimes x_i)x_{i+1}.$$

Applying σ to the relation (*) and substituting in the expression of $d(\sigma f)$, we obtain

$$d(\sigma f)(x_1 \otimes \ldots \otimes x_{i+1}) = x_1\,\sigma(f(x_2 \otimes \ldots \otimes x_{i+1})) - \sigma(x_1 f(x_2 \otimes \ldots \otimes x_{i+1}))$$
$$+(-1)^{i+1}[\sigma(f(x_1 \otimes \ldots \otimes x_i))\,x_{i+1} - \sigma(f(x_1 \otimes \ldots \otimes x_i)\,x_{i+1})]$$

for all $x_1, \ldots, x_{i+1}$ in P.

Notice that the second and fourth terms of this sum are often zero: indeed, f takes values in the simple Λ-bimodule $k\gamma$. But γ is an oriented path of length $n-2$ from, say s to t. Hence the only basis element with a non-zero action from the right is s, and from the left is t.

The first and third terms can only be non-zero if x_1 is an arrow or the vertex t and x_{i+1} is an arrow or the vertex s. Recall that $r^n = 0$, thus if x_1 (or x_{i+1}) is of length greater than two then $x_1\gamma = 0$ (or $\gamma x_{i+1} = 0$), because the length of γ is $n-2$.

If $x_1 = t$, the value of the first term is equal to the value of the second term. If $x_{i+1} = s$, the second is equal to the fourth.

With these facts in mind, we can conclude the following:

$$d(\sigma f)(x_1 \otimes \ldots \otimes x_{i+1}) = 0$$

if both x_1 and x_{i+1} are not arrows,

$$= x_1\,\sigma(f(x_2 \otimes \ldots \otimes x_{i+1})) + (-1)^{i+1}\sigma(f(x_1 \otimes \ldots \otimes x_i))\,x_{i+1}$$

if both x_1 and x_{i+1} are arrows,

$$= x_1\,\sigma(f(x_2 \otimes \ldots \otimes x_{i+1}))$$

if x_1 is an arrow but not x_{i+1},

$$= (-1)^{i+1}\sigma(f(x_1 \otimes \ldots \otimes x_1))\,x_{i+1}$$

if x_{i+1} is an arrow but not x_1.

This description of δf on the basis elements of $\Lambda^{\otimes i+1}$ coincides with the description of $\mathscr{S} \cup f + (-1)^{i+1} f \cup \mathscr{S}$, directly as cochains. In fact,

$$(\mathscr{S} \cup f)(x_1 \otimes \ldots \otimes x_{i+1}) = \sum_{a \in Q_1} f_a(x_1)(\sigma f)(x_2 \otimes \ldots \otimes x_{i+1})$$
$$= \begin{cases} x_1(\sigma f)(x_2 \otimes \ldots \otimes x_{i+1}) & \text{if } x_1 \text{ is an arrow,} \\ 0 & \text{otherwise.} \end{cases}$$

(Recall that $f_a(a) = a$ and $f_a(e) = 0$ for all $e \in P$, $e \neq a$.)

We return now to the case of 2-nilpotent algebras, and give a description of $H^2(\Lambda, \Lambda/r)$ and $H^3(\Lambda, r)$. We assume that the quiver Q does not contain loops or unoriented triangles. We could give a description without these assumptions, but it is simpler to work with them.

Let E be the set of oriented paths of length two with same source and end vertex. More precisely $E = \coprod_{s,t \in Q_0} (sAt) \times (tAs)$, since Q has no loops.

For each arrow a from s to t, let L_a be the set of length-three oriented paths from s to t, and $L = \coprod_{a \in A} L_a$.

We consider the vector spaces kE and kL with bases E and L. Define a linear map $D: kE \to kL$ in the following way: for each $(b, a) \in (sAt) \times (tAs) \subset E$,

$$D(b,a) = \sum_{x \in As} (xba)_x - \sum_{y \in sA} (bay)_y,$$

where the index arrow means that $(xba)_x \in L_x$; if there is a parallel arrow x' to x (same source, same end) we have to make a distinction between elements of L_x and $L_{x'}$.

The D we have just defined is the right model for δ, as we see in the following.

LEMMA 3. *There is an isomorphism between $H^2(\Lambda, \Lambda/r)$ and kE, and between $H^3(\Lambda, r)$ and kL such that the diagram*

$$\begin{array}{ccc} H^2(\Lambda, \Lambda/r) & \xrightarrow{\delta} & H^3(\Lambda, r) \\ \downarrow & & \downarrow \\ kE & \xrightarrow{D} & kL \end{array}$$

commutes.

Proof. Since $\Lambda/r = \oplus_{s \in Q_0} ks$ as bimodules, $H^2(\Lambda, \Lambda/r) = \oplus_{s \in Q_0} H^2(\Lambda, ks)$.

Let $(b, a) \in (sAt) \times (tAs)$, and consider $f_{(b,a)}: \Lambda \otimes \Lambda \to ks$ defined by

$$f_{(b,a)}(b \otimes a) = s$$

and $f_{(b,a)}$ gives zero when evaluated in any other tensor product of two basis vectors of Λ. These linear maps are easily checked to be 2-cocycles and they are linearly independent modulo coboundaries: if $g: \Lambda \to ks$ is a linear map, then

$$(dg)(b \otimes a) = bg(a) - g(ba) + g(b)a = 0$$

because each term in the sum is zero (ks is a semisimple bimodule and $ba \in r^2 = 0$); thus if $\sum \lambda_{ba} f_{(b,a)} = dg$, we obtain $\lambda_{ba} = 0$ for each $(b, a) \in sAt \times tAs$.

Moreover, the set $\{f_{(b,a)}\}_{(b,a) \in E}$ is a basis of $H^2(\Lambda, ks)$ because its cardinality is the dimension of $H^2(\Lambda, ks)$. Indeed,

$$H^2(\Lambda, ks) = H^2(\Lambda, \mathrm{Hom}_k(S, S)) = \mathrm{Ext}^2_\Lambda(S, S);$$

the dimension of $\mathrm{Ext}^2_\Lambda(S, S)$ is the number of paths of length two from s to s by Bongartz's result [**1**, p. 463].

Thus the left vertical isomorphism is obtained by sending $f_{(b,a)}$ to (b, a).

To obtain the one on the right consider $r = \oplus_{a \in A} ka$ as a bimodule. Hence $H^3(\Lambda, r) = \oplus_{a \in A} H^3(\Lambda, ka)$.

Let $(c_3 c_2 c_1)_a \in L_a$ be a path of length three from the source s of a to its end t. Define $f^a_{c_3 c_2 c_1}: \Lambda^{\otimes 3} \to ka$ by $f^a_{c_3 c_2 c_1}(c_3 \otimes c_2 \otimes c_1) = a$ and to be zero evaluated in any other tensor product of three basis elements of Λ. These linear maps are 3-cocycles, and linearly independent modulo coboundaries: if $g: \Lambda \otimes \Lambda \to ka$ is a linear map,

$$\begin{aligned} dg(c_3 \otimes c_2 \otimes c_1) &= c_3 g(c_2 \otimes c_1) - g(c_3 c_2 \otimes c_1) + g(c_3 \otimes c_2 c_1) - g(c_3 \otimes c_2) c_1 \\ &= 0, \end{aligned}$$

again because ka is a semisimple bimodule and $r^2 = 0$. Thus if a k-linear combination of the $f^a_{c_3 c_2 c_1}$ is equal to a coboundary, then all the coefficients are zero.

The $f^a_{c_3 c_2 c_1}$ are in the right number to be a basis of $H^3(\Lambda, ka)$, because

$$\dim_k H^3(\Lambda, ka) = \dim_k \mathrm{Ext}^3_\Lambda(S, T) = \dim_k t(F^3/F^4)s$$

(the last equality is derived from [**1**, p. 462, last line] with $n = 1$, $J = F$, $I = F^2$). But $\dim_k t(F^3/F^4)s$ is the number of length-three oriented paths from s to t, which is $|L_a|$.

The isomorphism between $H^3(\Lambda, r)$ and kL is obtained sending $f^a_{c_3 c_2 c_1}$ to $(c_3 c_2 c_1)_a$.

By composing these isomorphisms and δ, and using the description of δ in terms of cup products, we get exactly the defined D. We need only check that

$$f_x \cup f_{(b,a)} = f^x_{xba},$$

which is already true at the level of cochains.

We can now return to the claim which proves the theorem. Namely $D: kE \to kL$ is injective when Q is connected, does not contain loops or unoriented triangles, and Q is not the quiver $\bullet \rightleftarrows \bullet$.

Suppose that $\sum_{(b,a)\in E} \lambda_{(b,a)} D(b,a) = 0$. Recall that

$$D(b,a) = \sum_{x \in As} (xba)_x - \sum_{y \in sA} (bay)_y$$

if $(b,a) \in sAt \times tAs$.

The central remark is the following: for each (b,a) there exist $x \in As$, $x \neq a$ (or $y \in sA$, $y \neq b$). Indeed, Q is not the quiver $\bullet \rightleftarrows \bullet$ and Q is connected. Thus $(xba)_x$ (or $(bay)_y$) is a component of $D(b,a)$, but not a component of any other $D(d,c)$: if it were, then $(xba)_x = (zdc)_z$ or $(xba)_x = (dcz)_z$ for some arrow z. The first equality is impossible since we suppose that $(b,a) \neq (d,c)$. The second would imply that $x = z = a$, and so $x = a$, which is a contradiction. The argument is the same if $(bay)_y$ is a component of $D(b,a)$.

Using this remark we see that replacing each $D(b,a)$ by its value in the linear dependence relation, we get a linear combination of the basis L, such that for each $(b,a) \in E$ there exists x (or y) with the coefficient of $(xba)_x$ (or $(bay)_y$) equal to $\lambda_{(b,a)}$. Hence $\lambda_{(b,a)} = 0$ and D is injective.

In contrast with this, the matrix of D for $Q = \bullet \rightleftarrows \bullet$ *is*

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and so $\dim_k H^2(kQ/F^2, kQ/F^2) = 1$ in this case.

We conclude this note with a remark.

In [3] we gave sufficient conditions on a basic and connected finite dimensional algebra Λ for the vanishing of $H^2(\Lambda, \Lambda)$. Let Q be a connected quiver, let I be an admissible ideal of kQ, and let $\Lambda = kQ/I$. If for all couples of vertices (s, t) such that $t\Lambda s \neq 0$ we have $tIs = t(FI+IF)s$ then $H^2(\Lambda, \Lambda) = 0$.

These conditions are not necessary: when

$$Q = \bullet \rightleftarrows \bullet \longrightarrow \bullet, \quad I = F^2, \quad \text{and} \quad \Lambda = kQ/F^2,$$

we have $s\Lambda s \neq 0$, and $sF^2s \neq sF^3s$. At the same time we know from this note that $H^2(\Lambda, \Lambda) = 0$.

But if Q has no oriented cycles and $I = F^2$, the conditions of [3] are equivalent to the fact that Q does not contain unoriented triangles. From the above theorem we know that this is necessary and sufficient for $H^2(kQ/F^2, kQ/F^2)$ to be zero, when Q has no oriented cycles.

References

1. K. Bongartz, 'Algebras and quadratic forms', *J. London Math. Soc.* (2) 28 (1983) 461–469.
2. H. Cartan and S. Eilenberg, *Homological algebra* (University Press, Princeton, 1956).
3. C. Cibils, 'Cohomologie de Hochschild d'algebres de dimension finie', *Comm. Algebra* to appear.
4. P. Gabriel, 'Indecomposable representations II', Symposia Matematica 2 (Istituto Nazionale di alta Matematica, Roma, 1973) 81–104.
5. P. Gabriel, 'Auslander–Reiten sequences and representation finite algebras', *Representation theory* I, *proceedings*, 1979, Lecture Notes in Mathematics 831 (Springer, Berlin, 1980) 1–71.
6. P. Gabriel, 'Finite representation type is open', *Representations of algebras*, Ottawa 1974, Lecture Notes in Mathematics 488 (Springer, Berlin, 1975) 132–155.
7. M. Gerstenhaber, 'On the deformations of rings and algebras', *Ann. of Math.* 79 (1964) 59–103.

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