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A Rate for the Erdős–Turán Law*

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For Paul Erdős on his 80th birthday

The Erdős–Turán law gives a normal approximation for the order of a randomly chosen permutation of n objects. In this paper, we provide a sharp error estimate for the approximation, showing that, if the mean of the approximating normal distribution is slightly adjusted, the error is of order $\log^{-1/2} n$.

1. Introduction

Let σ denote a permutation of n objects, and $O(\sigma)$ its order. Landau [13] proved that $\max_\sigma \log O(\sigma) \sim \{n \log n\}^{1/2}$. In contrast, if σ is a single cycle of length n, $\log O(\sigma) = \log n$, such cycles constituting a fraction $1/n$ of all possible σ's. In view of the wide discrepancy between these extremes, the lovely theorem of Erdős and Turán (1967) comes as something of a surprise: that, for any x,

$$\frac{1}{n!} \# \{\sigma : \log O(\sigma) < \tfrac{1}{2}\log^2 n + x\{\tfrac{1}{3}\log^3 n\}^{1/2}\} \sim \Phi(x),$$

where Φ denotes the standard normal distribution function. In probabilistic terms, their result is expressed as

$$\mathbb{P}[\{\tfrac{1}{3}\log^3 n\}^{-1/2}(\log O(\sigma) - \tfrac{1}{2}\log^2 n) < x] \sim \Phi(x), \tag{1.1}$$

with σ now thought of as a permutation chosen at random, each of the $n!$ possibilities being equally likely. They remark that

'Our proof is a direct one and rather long; but a *first* proof can be as long as it wants to be. It would be however of interest to deduce it from the general principles of probability theory.'

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They also entertain hopes of finding a sharp remainder for their approximation.

Shorter probabilistic proofs of (1.1) are given by [5], [6] and [1], the last exploiting the Feller coupling to a record value process. Stein (unpublished) gives another coupling proof, with an error estimate of order $\log^{-1/4} n\{\log\log n\}^{1/2}$, which he describes as 'rather poor'. In fact, [16] sharpens the approach of Erdős and Turán, showing that the first correction to (1.1) is a mean shift of $-\log n \log\log n$, and that the error then remaining is of order at most $O(\log^{-1/2} n \log\log\log n)$. Nicolas also conjectures that the iterated logarithm in the error is superfluous. Our birthday present is to show this, by probabilistic means, not only for the uniform distribution on the set of permutations, but also under any Ewens sampling distribution. Since many combinatorial structures are, in a suitable sense, very closely approximated by one of the Ewens sampling distributions (see [4]), the result carries over easily to many other contexts. A typical example is the l.c.m. of the degrees of the factors of a random polynomial over the finite field with q elements, thus improving upon a theorem of [15].

Consider the probability measure μ_θ on the permutations of n objects determined by

$$\mu_\theta(\sigma) = \frac{\theta^{k(\sigma)}}{\theta_{(n)}}, \tag{1.2}$$

where $k(\sigma)$ is the number of cycles in σ, $\theta > 0$ is a parameter that can be chosen at will, and where rising factorials are denoted by

$$x_{(n)} = x(x+1)\dots(x+n-1), \quad x_{(0)} = 1.$$

If $\theta = 1$, the uniform distribution is recovered. Under μ_θ, the probability of the set of permutations having a_j cycles of length j, $1 \leqslant j \leqslant n$, is given by

$$I\left\{\sum_{j=1}^{n} ja_j = n\right\} \frac{n!}{\theta_{(n)}} \prod_{j=1}^{n} \left(\frac{\theta}{j}\right)^{a_j} \frac{1}{a_j!}, \tag{1.3}$$

as may be verified by multiplying the probability (1.2) by the number of permutations that have the given cycle index.

The joint distribution of cycle counts given by (1.3) is known as the Ewens sampling formula with parameter θ. It was derived by Ewens [8] in the context of population genetics. Ewens [9] provides an account of this theory that is accessible to mathematicians.

Under the Ewens sampling formula, the joint distribution of the cycle counts converges to that of independent Poisson random variables with mean θ/i as $n \to \infty$. Indeed, using the Feller coupling, the cycle counts for all values of n can be linked simultaneously on a common probability space with a single set of independent Poisson random variables with the appropriate means. The following precise statement of this fact comes essentially from [2].

Proposition 1.1. *Let $\{\xi_j, j \geqslant 1\}$ be a sequence of independent Bernoulli random variables satisfying*

$$\mathbb{P}[\xi_j = 1] = \frac{\theta}{\theta + j - 1}. \tag{1.4}$$

Define $(Z_{jm}, j \geqslant 1)$ *by*

$$Z_{jm} = \sum_{i=m+1}^{\infty} \xi_i(1-\xi_{i+1})\dots(1-\xi_{i+j-1})\,\xi_{i+j}, \tag{1.5}$$

and set $Z_j \equiv Z_{j0}$ *and* $Z = (Z_j, j \geqslant 1)$. *Define* $C^{(n)} = (C_j(n), j \geqslant 1)$ *by*

$$\begin{aligned} C_j(n) &= \sum_{i=1}^{n-j} \xi_i(1-\xi_{i+1})\dots(1-\xi_{i+j-1})\,\xi_{i+j} + \xi_{n-j+1}(1-\xi_{n-j+2})\dots(1-\xi_n) \\ &= Z_j - Z_{j,\,n-j} + \xi_{n-j+1}(1-\xi_{n-j+2})\dots(1-\xi_n) \end{aligned} \tag{1.6}$$

for $1 \leqslant j \leqslant n$, *setting* $C_j(n) = 0$ *for* $j > n$. *Then* $\mathbb{P}[(C_1(n), \dots, C_n(n)) = (a_1, \dots, a_n)]$ *is given by* (1.3), *and the* Z_j *are independent Poisson random variables with* $\mathbb{E}Z_j = \theta/j$. *Furthermore, for* $j \geqslant 1$,

$$Z_j - Z_{jn} - I[J_n + K_n = j+1] \leqslant C_j(n) \leqslant Z_j + I[J_n = j], \tag{1.7}$$

where J_n *and* K_n *are defined by*

$$J_n = \min\{j \geqslant 1 : \xi_{n-j+1} = 1\} \quad \textit{and} \quad K_n = \min\{j \geqslant 1 : \xi_{n+j} = 1\}. \tag{1.8}$$

With this representation, the order of the random permutation is $O_n(C^{(n)})$, where, for any $a \in \mathbb{N}^\infty$,

$$O_n(a) = \text{l.c.m.}\{i : 1 \leqslant i \leqslant n, a_i > 0\} \leqslant P_n(a) = \prod_{i=1}^{n} i^{a_i}.$$

On the other hand, from (1.6), $C_j(n)$ is close to Z_j for each j when n is large, so $\log O_n(C^{(n)})$ might plausibly be well approximated by $\log O_n(Z)$. Now functions involving Z are very much easier to handle than are the same functions of $C^{(n)}$, because the components Z_j of Z are independent and have known distributions. In particular, $\log O_n(Z)$ is close enough for our purposes to $\log P_n(Z) - \theta \log n \log\log n$, and

$$\log P_n(Z) = \sum_{i=1}^{n} Z_i \log i$$

is just a sum of independent random variables. The classical Berry–Esséen theorem [10, p. 544, Theorem 2] can thus be invoked to determine the accuracy of the normal approximation to its distribution.

The above arguments, justified in detail in Section 2, lead to the following result.

Theorem 1.2. *If* $C^{(n)}$ *is distributed according to the Ewens sampling formula* (1.3) *with parameter* θ,

$$\sup_x \left| \mathbb{P}\left[\left\{ \frac{\theta}{3} \log^3 n \right\}^{-1/2} \left(\log O_n(C^{(n)}) - \frac{\theta}{2} \log^2 n + \theta \log n \log\log n \right) \leqslant x \right] - \Phi(x) \right| = O(\{\log n\}^{-1/2}).$$

It would not be difficult to give an explicit bound for the constant implied in the error term. Indeed, the leading contributions arise from a Berry–Esséen estimate, for which the necessary quantities are estimated in Proposition 2.4, from inequality (2.1), for which (2.2) and Lemma 2.5 already provide a bound, and from the next mean correction, which requires a more careful asymptotic evaluation following (2.4).

A process variant of Theorem 1.2 can also be formulated. Let W_n be the random element of $D[0,1]$ defined by

$$W_n(t) = \left\{\frac{\theta}{3}\log^3 n\right\}^{-1/2}\left(\log O_{[n^t]}(C^{(n)}) - \frac{\theta}{2}t^2\log^2 n\right).$$

Theorem 1.3. *It is possible to construct $C^{(n)}$ and a standard Brownian motion W on the same probability space, in such a way that*

$$\mathbb{E}\left\{\sup_{0\leqslant t\leqslant 1}|W_n(t) - W(t^3)|\right\} = O\left(\frac{\log\log n}{\sqrt{\log n}}\right).$$

2. Proofs

As previously indicated, the proof of Theorem 1.2 consists of showing that $\log O_n(C^{(n)})$ is close enough to $\log O_n(Z)$, which in turn is close enough to $\log P_n(Z) - \theta\log n\log\log n$. The Berry–Esséen theorem then gives the normal approximation for $\log P_n(Z)$.

For vectors a and b, define $|a-b| = \sum_i |a_i - b_i|$. Since $O_n(a) \leqslant O_n(b)\, n^{|b-a|}$ whenever a and b are vectors with $a \leqslant b$, it follows from (1.7) that

$$\log O_n(Z) - (Y_n + 1)\log n \leqslant \log O_n(C^{(n)}) \leqslant \log O_n(Z) + \log n, \tag{2.1}$$

where $Y_n = \sum_{j=1}^n Z_{jn}$ is independent of $C^{(n)}$, and

$$\begin{aligned}\mathbb{E}Y_n &= \sum_{j=1}^n \sum_{i>n}\left(\frac{\theta}{\theta+i-1}\right)\left(\frac{\theta}{\theta+i+j-1}\right)\prod_{l=i+1}^{i+j-1}\left(\frac{l-1}{\theta+l-1}\right) \\ &\leqslant \theta^2\sum_{j=1}^n\sum_{i>n}\left(\frac{1}{i-1}\right)\left(\frac{1}{i+j-1}\right) \leqslant \theta^2.\end{aligned} \tag{2.2}$$

Inequality (2.1) combined with (2.2) is enough for the closeness of $\log O_n(C^{(n)})$ and $\log O_n(Z)$.

Next, we can compute the difference between $\log O_n(Z)$ and $\log P_n(Z)$ using a formula of [5] and [14]:

$$\log P_n(Z) - \log O_n(Z) = {\sum_p}' \sum_{s\geqslant 1}(D_{np^s} - 1)^+ \log p, \tag{2.3}$$

where $\sum'$ and $\sum''$ denote sums over *prime* indices, and

$$D_{nk} = \sum_{j\leqslant n:\, k|j} Z_j.$$

Considering first its expectation, observe that, since $(d-1)^+ = d - 1 + I\{d=0\}$,

$$\begin{aligned}\mathbb{E}(D_{nk}-1)^+ &= \mathbb{E}D_{nk} - 1 + \mathbb{P}[D_{nk}=0] \\ &= \lambda_{nk} - 1 + e^{-\lambda_{nk}} \leqslant (\lambda_{nk} \wedge \tfrac{1}{2}\lambda_{nk}^2),\end{aligned} \tag{2.4}$$

where

$$\lambda_{nk} = \sum_{j \leqslant n : k | j} j^{-1}\theta = \begin{cases} k^{-1}\theta\psi([n/k]+1) & \text{if } k \leqslant n; \\ 0 & \text{if } k > n, \end{cases}$$

and $\psi(r+1) = \sum_{j=1}^{r} j^{-1}$. Hence

$$\mu_n := \mathbb{E}\{\log P_n(Z) - \log O_n(Z)\} = \sum_p{}' \sum_{s \geqslant 1} \log p \mathbb{E}(D_{np^s} - 1)^+$$

$$= \sum_p{}' \sum_{s \geqslant 1} \log p(\lambda_{np^s} - 1 + \exp\{-\lambda_{np^s}\}) = \sum_{p \leqslant \log n}{}' \theta p^{-1} \log p \log n + O(\log n)$$

$$= \theta \log n \log\log n + O(\log n),$$

where the estimates use (2.4), integration by parts, and Theorems 7 and 425 of [11].

For the variability of $\log O_n(Z) - \log P_n(Z)$, we now need two lemmas.

Lemma 2.1. *For $p \neq q$ prime and $s, t \geqslant 1$,*

$$\operatorname{Cov}((D_{np^s} - 1)^+, (D_{nq^t} - 1)^+) \leqslant \frac{\theta(1 + \log n)}{p^s q^t}.$$

Proof. Set

$$\lambda_1 = \sum_{\substack{j=1 \\ p^s | j}}^{n} j^{-1}, \quad \lambda_2 = \sum_{\substack{i=1 \\ q^t | i}}^{n} i^{-1} \quad \text{and} \quad \xi = \sum_{\substack{l=1 \\ p^s q^t | l}}^{n} l^{-1} \leqslant \frac{(1 + \log n)}{p^s q^t},$$

and write $D_1 = D_{np^s}$ and $D_2 = D_{nq^t}$. Then, in the expansion

$$\operatorname{Cov}((D_1 - 1)^+, (D_2 - 1)^+) = \operatorname{Cov}(D_1, D_2) + \operatorname{Cov}(D_1, I[D_2 = 0]) + \operatorname{Cov}(I[D_1 = 0], D_2) + \operatorname{Cov}(I[D_1 = 0], I[D_2 = 0]),$$

the first contribution is evaluated as

$$\operatorname{Cov}(D_1, D_2) = \mathbb{E}\left\{\sum_{\substack{j=1 \\ p^s | j}}^{n} \sum_{\substack{i=1 \\ q^t | i}}^{n} (Z_j - j^{-1}\theta)(Z_i - i^{-1}\theta)\right\}$$

$$= \sum_{\substack{l=1 \\ p^s q^t | l}}^{n} \operatorname{Var} Z_l = \theta \sum_{\substack{l=1 \\ p^s q^t | l}}^{n} l^{-1} = \theta\xi,$$

because of the independence of the Z_j's. For the second contribution, we have

$$\operatorname{Cov}(D_1, I[D_2 = 0]) = \mathbb{P}\left[\bigcap_{\substack{i=1 \\ q^t | i}}^{n} \{Z_i = 0\}\right]\{\mathbb{E}(D_1 \mid D_2 = 0) - \mathbb{E}D_1\} = -\theta\xi e^{-\theta\lambda_2},$$

and similarly for the third, and for the last we have

$$\operatorname{Cov}(I[D_1 = 0], I[D_2 = 0]) = e^{-\theta(\lambda_1 + \lambda_2)}\{e^{\theta\xi} - 1\} \leqslant \theta\xi e^{-\theta(\lambda_1 + \lambda_2 - \xi)}.$$

Hence

$$\mathrm{Cov}\,((D_1-1)^+,(D_2-1)^+) = \theta\xi\{1-e^{-\theta\lambda_1}-e^{-\theta\lambda_2}+e^{-\theta(\lambda_1+\lambda_2-\xi)}\} \leqslant \theta\xi,$$

proving the lemma.

Lemma 2.2. *For* $1 \leqslant s \leqslant t$,

$$\mathrm{Cov}\,((D_{np^s}-1)^+,(D_{np^t}-1)^+) \leqslant \theta p^{-t}(1+\log n).$$

Proof. The argument runs as for Lemma 2.1, with λ_1 defined as before, but now with

$$\xi = \lambda_2 = \sum_{\substack{l=1\\ p^t|l}}^{n} l^{-1} \leqslant p^{-t}(1+\log n).$$

The computations now yield

$$\mathrm{Cov}\,(D_1,D_2) = \theta\xi; \quad \mathrm{Cov}\,(D_1, I[D_2=0]) = -\theta\xi e^{-\theta\lambda_2}; \quad \mathrm{Cov}\,(I[D_1=0],D_2) = -\theta\lambda_2 e^{-\theta\lambda_1}$$

and

$$\mathrm{Cov}\,(I[D_1=0], I[D_2=0]) = e^{-\theta\lambda_1}(1-e^{-\theta\lambda_2}),$$

and thus

$$\mathrm{Cov}\,((D_1-1)^+,(D_2-1)^+) = \theta\xi(1-e^{-\theta\lambda_2})+e^{-\theta\lambda_1}(1-e^{-\theta\lambda_2}-\theta\lambda_2) \leqslant \theta\xi(1-e^{-\theta\lambda_2}) \leqslant \theta\xi.$$

The two lemmas enable us to control the difference between $\log O_n(Z)$ and $\log P_n(Z)$ as follows.

Proposition 2.3. *For any* $K > 0$,

$$\mathbb{P}[|\log P_n(Z)-\log O_n(Z)-\mu_n| > K\log n] = O\left(\frac{(\log\log n)^2}{\log n}\right).$$

Proof. Write

$$\begin{aligned}\log P_n(Z)-\log O_n(Z) &= \left(\sum_{p\leqslant \log^2 n}{}' + \sum_{p>\log^2 n}{}'\right)(D_{np}-1)^+\log p + \sum_p{}' \sum_{s\geqslant 2}(D_{np^s}-1)^+\log p\\ &= V_1+V_2+V_3,\end{aligned}$$

say. Lemmas 2.1 and 2.2 give

$$\begin{aligned}\mathrm{Var}\,V_1 &\leqslant \sum_{p\leqslant \log^2 n}{}' \frac{\theta(1+\log n)}{p}\log^2 p + \sum_{p\neq q\leqslant \log^2 n}{}'' \frac{\theta(1+\log n)}{pq}\log p\log q\\ &= O(\log n(\log\log n)^2);\end{aligned}$$

it follows from (2.4) that

$$\mathbb{E}V_2 \leqslant \frac{\theta^2}{2}\sum_{p>\log^2 n}{}' p^{-2}\log p(1+\log n)^2 = O(1);$$

and Lemmas 2.1 and 2.2 imply that

$$\operatorname{Var} V_3 \leqslant \sum_p{}' \log^2 p \sum_{s,t \geqslant 2} \frac{\theta(1+\log n)}{p^{s \vee t}} + \sum_{p \neq q}{}'' \log p \log q \sum_{s,t \geqslant 2} \frac{\theta(1+\log n)}{p^s q^t} = O(\log n).$$

Thus, by Chebyshev's inequality,

$$\mathbb{P}[|V_1 - \mathbb{E}V_1| > \tfrac{1}{3}K\log n] = O(\log^{-1} n(\log\log n)^2);$$
$$\mathbb{P}[|V_2 - \mathbb{E}V_2| > \tfrac{1}{3}K\log n] = O(\log^{-1} n),$$

and

$$\mathbb{P}[|V_3 - \mathbb{E}V_3| > \tfrac{1}{3}K\log n] = O(\log^{-1} n),$$

proving the proposition.

We now use the closeness of the quantities $\log O_n(C^{(n)})$, $\log O_n(Z)$ and $\log P_n(Z) - \mu_n$ to prove Theorem 1.2. To do so, we introduce the standardized random variables

$$S_{1n} = \frac{\log P_n(Z) - \dfrac{\theta}{2}\log^2 n}{\sqrt{\dfrac{\theta}{3}\log^3 n}}; \quad S_{2n} = \frac{\log O_n(Z) + \mu_n - \dfrac{\theta}{2}\log^2 n}{\sqrt{\dfrac{\theta}{3}\log^3 n}},$$

and

$$S_{3n} = \frac{\log O_n(C^{(n)}) + \mu_n - \dfrac{\theta}{2}\log^2 n}{\sqrt{\dfrac{\theta}{3}\log^3 n}},$$

whose distributions we shall successively approximate. Since the quantity $\log P_n(Z)$ can be written in the form $\sum_{j=1}^n Z_j \log j$ as a weighted sum of independent Poisson random variables, the normal approximation for S_{1n} follows easily from the Berry–Esséen theorem.

Proposition 2.4. *There exists a constant $c_1 = c_1(\theta)$ such that*

$$\sup_x |\mathbb{P}[S_{1n} \leqslant x] - \Phi(x)| \leqslant c_1 \log^{-1/2} n.$$

Proof. It is enough to note that

$$\sum_{j=1}^n \mathbb{E}(Z_j \log j) = \theta \sum_{j=1}^n j^{-1} \log j = \frac{\theta}{2}(\log^2 n + O(1)),$$

that

$$\sum_{j=1}^n \operatorname{Var}(Z_j \log j) = \theta \sum_{j=1}^n j^{-1} \log^2 j = \frac{\theta}{3}(\log^3 n + O(1))$$

and that

$$\sum_{j=1}^n \mathbb{E}|Z_j - \mathbb{E}Z_j|^3 \log^3 j = O(\log^4 n):$$

indeed, for $j \geqslant \theta$,

$$\mathbb{E}|Z_j - \mathbb{E}Z_j|^3 = \frac{\theta}{j} + \frac{2\theta^3}{j^3} e^{-\theta/j} \leqslant \frac{\theta}{j}[1+2e^{-1}],$$

and hence, for $\theta \leqslant 2$,

$$\sum_{j=1}^{n} \mathbb{E}|Z_j - \mathbb{E}Z_j|^3 \log^3 j \leqslant \theta[1+2e^{-1}] \sum_{j=1}^{n} j^{-1} \log^3 j = \frac{\theta[1+2e^{-1}]}{4}(\log^4 n + O(1)). \tag{2.5}$$

In order to show that S_{2n} and S_{3n} have almost the same distribution as S_{1n}, because of Proposition 2.3 and (2.1), one further lemma is required.

Lemma 2.5. *Let U and X be random variables with $\sup_x |\mathbb{P}[U \leqslant x] - \Phi(x)| \leqslant \eta$. Then, for any $\epsilon > 0$,*

$$\sup_x |\mathbb{P}[U+X \leqslant x] - \Phi(x)| \leqslant \eta + \frac{\epsilon}{\sqrt{2\pi}} + \mathbb{P}[|X| > \epsilon]. \tag{2.6}$$

If W and Y are independent random variables with $\mathbb{E}Y < \infty$, and if $|W-U| \leqslant Y$, then

$$\sup_x |\mathbb{P}[W \leqslant x] - \Phi(x)| \leqslant 3\left\{\eta + \frac{4\mathbb{E}Y}{\sqrt{2\pi}}\right\}. \tag{2.7}$$

Proof. The first part is standard. For the second, let $\delta_y = \mathbb{P}[W \leqslant y] - \Phi(y)$ and set $\Delta = \sup_y |\delta_y|$. Write $\rho = 3\mathbb{E}Y$ and $p = \mathbb{P}[Y > \rho]$, so that $p \leqslant 1/3$. Then, since, for any x, $\{U \leqslant x\} \supset \{W+Y \leqslant x\}$, it follows that

$$\begin{aligned}\mathbb{P}[U \leqslant x] &\geqslant \int_{[0,\infty)} \mathbb{P}[W \leqslant x-y]\, F_Y(dy) \\ &\geqslant (1-p)\,\mathbb{P}[W \leqslant x-\rho] + \int_{(\rho,\infty)} \Phi(x-y)\, F_Y(dy) - p\Delta,\end{aligned}$$

where F_Y denotes the distribution function of Y. Hence, comparing as much as possible to $\Phi(x-\rho)$, it follows that

$$\Phi(x-\rho) + \eta + \frac{\rho}{\sqrt{2\pi}} \geqslant (1-p)\,\mathbb{P}[W \leqslant x-\rho] + p\Phi(x-\rho) - \frac{\mathbb{E}\{(Y-\rho)\, I[Y>\rho]\}}{\sqrt{2\pi}} - p\Delta,$$

implying that

$$(1-p)\,\delta_{x-\rho} \leqslant \eta + \frac{4\mathbb{E}Y}{\sqrt{2\pi}} + p\Delta.$$

A similar argument starting from $\{U \leqslant x\} \subset \{W - Y \leqslant x\}$ then gives

$$-(1-p)\,\delta_{x+\rho} \leqslant \eta + \frac{4\mathbb{E}Y}{\sqrt{2\pi}} + p\Delta.$$

The choice of x being arbitrary, it thus follows that

$$(1-p)\Delta \leqslant \eta + \frac{4\mathbb{E}Y}{\sqrt{2\pi}} + p\Delta$$

also, and hence that

$$\Delta \leqslant 3\left\{\eta + \frac{4\mathbb{E}Y}{\sqrt{2\pi}}\right\},$$

as claimed.

To complete the proof of Theorem 1.2, apply (2.6) with S_{1n} for U and $S_{2n}-S_{1n}$ for X, taking $\eta = c_1 \log^{-1/2} n$ from Proposition 2.4 and $\epsilon = \log^{-1/2} n$. By Proposition 2.3,

$$\mathbb{P}[|S_{2n}-S_{1n}| > \epsilon] = \mathbb{P}\left[|\log P_n(Z) - \log O_n(Z) - \mu_n| > \epsilon\sqrt{\frac{\theta}{3}\log^3 n}\right] = O\left(\frac{(\log\log n)^2}{\log n}\right),$$

and hence, from (2.6),

$$\sup_x |\mathbb{P}[S_{2n} \leqslant x] - \Phi(x)| \leqslant c_2 \log^{-1/2} n$$

for some $c_2 = c_2(\theta)$. Now we can apply (2.7) with $U = S_{2n}$ and $W = S_{3n}$, since (2.1) implies that $|U-W| \leqslant Y$, with $Y = \{(\theta/3)\log n\}^{-1/2}(Y_n+1)$, giving

$$\sup_x |\mathbb{P}[S_{3n} \leqslant x] - \Phi(x)| = O(\log^{-1/2} n(1+\mathbb{E}Y_n)) = O(\log^{-1/2} n),$$

in view of (2.2). This is equivalent to Theorem 1.2.

To prove Theorem 1.3, we use essentially the same estimates. First, from (2.1),

$$|\log O_{[n^t]}(C^{(n)}) - \log O_{[n^t]}(Z)| \leqslant (1+Y_n)\log n$$

for all $0 \leqslant t \leqslant 1$, and then, from (2.3),

$$\begin{aligned} 0 &\leqslant \log P_{[n^t]}(Z) - \log O_{[n^t]}(Z) \\ &= \sum_p{}' \sum_{s\geqslant 1} (D_{[n^t]p^s} - 1)^+ \log p \leqslant \sum_p{}' \sum_{s\geqslant 1} (D_{np^s} - 1)^+ \log p. \end{aligned}$$

Hence

$$\mathbb{E}\left\{\sup_{0\leqslant t\leqslant 1} |\log O_{[n^t]}(C^{(n)}) - \log P_{[n^t]}(Z)|\right\} = O(\log\log n \log n).$$

Now

$$\log P_{[n^t]}(Z) = \sum_{j=1}^{[n^t]} Z_j \log j = \sum_{j=1}^{[n^t]} j^{-1}\theta\log j + \sqrt{\theta\psi(n+1)}\int_0^t s\log n\, dB_n(s),$$

where

$$B_n(t) = \sum_{j=1}^{[n^t]}\left(\frac{Z_j - j^{-1}\theta}{\sqrt{\theta\psi(n+1)}}\right)$$

can be realized as

$$\{\theta\psi(n+1)\}^{-1/2}\{P(\theta\psi([n^t]+1))-\theta\psi([n^t]+1)\}$$

using a Poisson process P with unit rate. Also, since

$$\left|\int_0^t s[dB_n(s)-dB(s)]\right| = \left|t[B_n(t)-B(t)]-\int_0^t \{B_n(s)-B(s)\}\,ds\right| \leqslant 2 \sup_{0\leqslant t\leqslant 1} |B_n(t)-B(t)|,$$

the uniform approximation of B_n by a standard Brownian motion B, in the form

$$\mathbb{E}\left\{\sup_{0\leqslant t\leqslant 1} |B_n(t)-B(t)|\right\} = O(\{\log n\}^{-1/2}\log\log n),$$

as carried out using the theorem of Komlós, Major and Tusnády [12] in the case $\theta = 1$ in [3], now implies the conclusion of Theorem 1.3: take $W(t^3) = \sqrt{3}\int_0^t s\,dB_n(s)$.

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