IMA Journal of Numerical Analysis (1990) **10**, 171–180

A Half-Explicit Extrapolation Method for Differential-Algebraic Systems of Index 3

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[Received 1 May 1989]

The present paper is concerned with the numerical solution of differential-algebraic index 3 problems. We study an extrapolation method based on a variant of Euler's rule. This method only treats the algebraic variables implicitly, hence has computational advantages compared to fully implicit schemes. A numerical example in line with the theoretical results is included.

1. Introduction

IT is well known that extrapolation methods are an effective way to find the numerical solution of nonstiff and stiff differential equations with high accuracy, see Deuflhard (1985), Hairer, Nørsett, & Wanner (1987), Hairer & Lubich (1988), among others. In a series of papers several authors investigated the feasibility of extrapolation for the solution of differential-algebraic systems (Deuflhard, Hairer, & Zugck, 1987; Hairer & Lubich, 1987; Hairer, Lubich, & Roche, 1989; Lubich, 1989a). Yet, all methods considered in these articles were implicit or linearly implicit at least. Recently, Lubich (1989b) showed that Gragg's algorithm can be extended to index 2 problems in a quite direct way. The method retains the h^2 expansion of the error and only treats the algebraic components implicitly. A further extension to index 3 systems, however, does not seem to be possible without losing the h^2 expansion. We shall therefore generalize his ideas in another direction.

In the present article we propose an extrapolation method for index 3 problems that handles the differential components explicitly and only treats the algebraic variables implicitly, thus reducing the linear algebra considerably. In Section 2 we specify the class of differential-algebraic equations considered in this paper. We continue with a description of our basic method—a half-explicit Euler scheme—in Section 3. There we also prove the existence of an asymptotic expansion of the error, which gives the theoretical justification for extrapolation. In Section 4 we analyse the local error as well as the order of convergence for the resulting extrapolation method. A numerical example which is in line with the theoretical results is presented finally in Section 5.

It is worth mentioning that for the problems considered in this article the trapezoidal rule retains its h^2 expansion of the error for the differential components, but is only $O(H)$ in the algebraic variables. The fully implicit midpoint rule, on the other hand, has no expansion at all.

2. Description of the problems to be studied

This paper treats the numerical solution of differential-algebraic systems of the form

$$y' = f(y, z), \qquad z' = k(y, z, u), \qquad 0 = g(y), \tag{1}$$

where f, k, and g have sufficiently many bounded derivatives. The system (1) is supposed to be of index 3 (in the sense of either Gear, Leimkuhler, & Gupta (1985) or of Hairer, Lubich, & Roche (1989)), i.e.

the matrix $g_y(y)f_z(y, z)k_u(y, z, u)$ has a bounded inverse in a neighbourhood of the solution. (2a)

(Here and in the following, partial derivatives are denoted by corresponding subscripts.)

Further it is assumed that k is *linear* in u, i.e.

$$k(y, z, u) = k_0(y, z) + k_u(y, z)u. \tag{2b}$$

If one seeks a solution of (1) with an Euler-type method, this linearity assumption seems natural as it guarantees the existence of a solution for the implicit Euler discretization. (This subject is discussed in a remark following Theorem 6.1 in Hairer, Lubich, & Roche (1989).)

We suppose that the given initial values for the y and z components, y_0 and z_0 respectively, are consistent with (1), i.e.

$$g(y_0) = 0, \qquad g_y(y_0)f(y_0, z_0) = 0. \tag{2c}$$

Due to (2a, b), the initial value for the u component is uniquely determined by the relation

$$g_{yy}(f, f) + g_y f_y f + g_y f_z k = 0$$

at the point (y_0, z_0, u_0). Anway, it will not enter the numerical method.

Examples for such index 3 problems are holonomic mechanical systems. As u normally plays the role of a Lagrange multiplier, its linear appearance is justified. For more details the interested reader is referred to Hairer, Lubich, & Roche (1989) and the literature cited there (e.g. Gear, Leimkuhler, & Gupta (1985) or Gear & Petzold (1984)).

3. *h* extrapolation for the half-explicit Euler rule

For the numerical solution of (1) we consider the following half-explicit Euler rule

$$y_{n+1} = y_n + hf(y_n, z_{n+1}), \tag{3a}$$

$$z_{n+1} = z_n + hk(y_n, z_n, u_{n+1}), \tag{3b}$$

$$0 = g(y_{n+1}), \tag{3c}$$

for $n = 0, 1, 2, \ldots$.

The following theorem shows existence and uniqueness of the solution of (3) under appropriate assumptions.

THEOREM 1 *Let (2) be satisfied in an h-independent neighbourhood of* (η, ζ) *and assume that*

$$g(\eta + hf(\eta, \zeta)) = O(h^2).$$

Then the nonlinear system

$$y = \eta + hf(\eta, z), \tag{4a}$$

$$z = \zeta + hk(\eta, \zeta, u), \tag{4b}$$

$$0 = g(y) \tag{4c}$$

possesses a unique solution for $h \leq h_0$, *with* $u = O(1)$.

Proof. Inserting (4a,b) into (4c) yields a nonlinear system for u

$$g\big(\eta + hf(\eta, \hat{\zeta} + hk_u(\eta, \zeta)u)\big) = 0, \tag{5}$$

where $\hat{\zeta} = \zeta + hk_0(\eta, \zeta)$. To study this equation, it is convenient to consider the homotopy (as it is done in a similar way in Hairer, Lubich, & Roche (1989))

$$g\big(\eta + hf(\eta, \hat{\zeta} + k_u(\eta, \zeta)v)\big) + (\tau - 1)g(\eta + hf(\eta, \hat{\zeta})) = 0. \tag{6}$$

For $\tau = 0$ a solution is given by $v = 0$. Note that for $\tau = 1$, (6) is equivalent to (5) with $v = hu$. We consider v as a function of τ and differentiate (6) with respect to τ. This yields

$$hg_y\big(\eta + hf(\eta, \hat{\zeta} + k_u(\eta, \zeta)v)\big)f_z(\eta, \hat{\zeta} + k_u(\eta, \zeta)v)k_u(\eta, \zeta)\dot{v} + g(\eta + hf(\eta, \hat{\zeta})) = 0.$$

By assumption the second term is $O(h^2)$, yielding the differential equation

$$g_y\big(\eta + hf(\eta, \hat{\zeta} + k_u(\eta, \zeta)v)\big)f_z(\eta, \hat{\zeta} + k_u(\eta, \zeta)v)k_u(\eta, \zeta)\dot{v} = O(h).$$

Provided that v remains in an h-independent neighbourhood of 0, one has

$$\dot{v} = O(h)$$

and therefore

$$v(\tau) = \int_0^\tau \dot{v}(t)\,dt = O(h) \quad \text{for } |\tau| \leq 1.$$

Hence the differential equation possesses a solution at least for $0 \leq \tau \leq 1$ and

$$hu = v(1) = O(h).$$

To prove uniqueness, we take a second solution $\hat{u}$ of (5) and denote the difference by $\Delta u = u - \hat{u}$. Linearizing

$$0 = g\big(\eta + hf(\eta, \hat{\zeta} + hk_u(\eta, \zeta)u)\big) - g\big(\eta + hf(\eta, \hat{\zeta} + hk_u(\eta, \zeta)\hat{u})\big)$$

immediately gives

$$h^2 g_y(\eta + hf(\eta, \hat{\zeta}))f_z(\eta, \hat{\zeta})k_u(\eta, \zeta)\Delta u = O(h^3 \Delta u)$$

and hence $\Delta u = 0$. Reinserting the unique solution u into (4b) and (4a) gives z and y explicitly. □

Remark. For the important special case—where f is linear in z—equation (5) can be solved by simplified Newton iterations with the h-independent matrix $(g_y f_z k_u)(y_0, z_0)$ which is invertible by (2a).

Next, it will be shown that method (3) is well suited for h extrapolation.

THEOREM 2 *Under assumption* (2) *the method* (3) *is well defined for problem* (1) *for h small enough and its error possesses an h expansion of the form*

$$
\begin{aligned}
y_n - y(x_n) &= h a_1(x_n) + \cdots + h^N a_N(x_n) + O(h^{N+1}),\\
z_n - z(x_n) &= h\big(b_1(x_n) + \beta_n^1\big) + \cdots + h^N\big(b_N(x_n) + \beta_n^N\big) + O(h^{N+1}),\\
u_n - u(x_n) &= \gamma_n^0 + h\big(c_1(x_n) + \gamma_n^1\big) + \cdots + h^{N-1}\big(c_{N-1}(x_n) + \gamma_n^{N-1}\big) + O(h^N),
\end{aligned}
$$

valid for $0 \leqslant nh \leqslant \text{const}$, *with* $a_i(x_0) = 0$, $\beta_n^i = 0$ *for* $n \geqslant 1$, *and* $\gamma_n^i = 0$ *for* $n \geqslant 2$.

Proof. For the proof of Theorem 2 we shall use the ideas of Lubich (1989b) and Hairer, Lubich, & Roche (1989). The proof consists of two parts. First we will construct truncated expansions

$$
\begin{aligned}
\hat{y}_n &= y(x_n) + h a_1(x_n) + \cdots + h^{N+1} a_{N+1}(x_n),\\
\hat{z}_n &= z(x_n) + h\big(b_1(x_n) + \beta_n^1\big) + \cdots + h^{N+1}\big(b_{N+1}(x_n) + \beta_n^{N+1}\big),\\
\hat{u}_n &= u(x_n) + \gamma_n^0 + h\big(c_1(x_n) + \gamma_n^1\big) + \cdots + h^N\big(c_N(x_n) + \gamma_n^N\big),
\end{aligned}
$$

such that $\hat{y}_0 = y_0$, $\hat{z}_0 = z_0$, and the defect of $\hat{y}_{n+1}$, $\hat{z}_{n+1}$, $\hat{u}_{n+1}$ inserted into (3) is

$$
\begin{aligned}
\hat{y}_{n+1} &= \hat{y}_n + hf(\hat{y}_n, \hat{z}_{n+1}) + O(h^{N+3}),\\
\hat{z}_{n+1} &= \hat{z}_n + hk(\hat{y}_n, \hat{z}_n, \hat{u}_{n+1}) + O(h^{N+2}),\\
0 &= g(\hat{y}_{n+1}) + O(h^{N+2}).
\end{aligned}
\tag{7}
$$

The theorem then follows from a stability estimate given in (b).

(a) Truncated expansions: Inserting $\hat{y}_{n+1}$, $\hat{z}_{n+1}$, $\hat{u}_{n+1}$, $\hat{y}_n$, and $\hat{z}_n$ into (3) and developing into Taylor series shows (writing $f_y(x_n)$ instead of $f_y\big(y(x_n), z(x_n)\big)$, etc.)

$$
\begin{aligned}
&\hat{y}_{n+1} - \hat{y}_n - hf(\hat{y}_n, \hat{z}_{n+1}) =\\
&h^2[\tfrac{1}{2}y''(x_n) + a_1'(x_n) - f_y(x_n)a_1(x_n) - f_z(x_n)b_1(x_n) - f_z(x_n)z'(x_n)]\\
&\quad + h^3\big[\tfrac{1}{6}y'''(x_n) + \tfrac{1}{2}a_1''(x_n) + a_2'(x_n) - f_y(x_n)a_2(x_n) - f_z(x_n)b_2(x_n)\\
&\qquad - \tfrac{1}{2}f_z(x_n)z''(x_n) - f_z(x_n)b_1'(x_n) - \tfrac{1}{2}f_{yy}(x_n)\big(a_1(x_n)\big)^2\\
&\qquad - f_{yz}(x_n)a_1(x_n)\big(z'(x_n) + b_1(x_n)\big) - \tfrac{1}{2}f_{zz}(x_n)\big(z'(x_n) + b_1(x_n)\big)^2\big] + O(h^4),
\end{aligned}
$$

$$
\begin{aligned}
&\hat{z}_{n+1} - \hat{z}_n - hk(\hat{y}_n, \hat{z}_n, \hat{u}_{n+1}) = h\delta_{n0}[-\beta_0^1 - k_u(x_0)\gamma_1^0]\\
&+ h^2[\tfrac{1}{2}z''(x_n) + b_1'(x_n) - k_y(x_n)a_1(x_n) - k_z(x_n)b_1(x_n) - k_u(x_n)c_1(x_n) - k_u(x_n)u'(x_n)]\\
&+ h^2\delta_{n0}[-\beta_0^2 - k_u(x_0)\gamma_1^1 - k_z(x_0)\beta_0^1 - k_{yu}(x_0)a_1(x_0)\gamma_1^0 - k_{zu}(x_0)\big(b_1(x_0) + \beta_0^1\big)\gamma_0^1]\\
&+ h^3[b_2'(x_n) - k_y(x_n)a_2(x_n) - k_z(x_n)b_2(x_n) - k_u(x_n)c_2(x_n) - e_2(x_n)]\\
&+ h^3\delta_{n0}[-\beta_0^3 - k_u(x_0)\gamma_1^2 + \varepsilon_2] + O(h^4),
\end{aligned}
$$

$$
\begin{aligned}
g(\hat{y}_{n+1}) &= hg_y(x_{n+1})a_1(x_{n+1})\\
&\quad + h^2\big[g_y(x_{n+1})a_2(x_{n+1}) + \tfrac{1}{2}g_{yy}(x_{n+1})\big(a_1(x_{n+1})\big)^2\big] + O(h^3),
\end{aligned}
$$

with the Kronecker symbol δ_{n0} (which is 1 for $n=0$ and 0 otherwise). One obtains (7) for $N=0$ if

$$\begin{aligned} a_1'(x) &= f_y(x)a_1(x) + f_z(x)b_1(x) + f_z(x)z'(x) - \tfrac{1}{2}y''(x), \\ b_1'(x) &= k_y(x)a_1(x) + k_z(x)b_1(x) + k_u(x)c_1(x) + k_u(x)u'(x) - \tfrac{1}{2}z''(x), \\ 0 &= g_y(x)a_1(x), \end{aligned} \tag{8}$$

$$k_u(x_0)\gamma_1^0 = -\beta_0^1. \tag{9}$$

After adding the component $x'=1$, (8) is an index 3 system of the form (1). As in Hairer, Lubich, & Roche (1989) it is convenient to work with the projections

$$Q = k_u(x_0)[g_y(x_0)f_z(x_0)k_u(x_0)]^{-1}g_y(x_0)f_z(x_0) \quad \text{and} \quad P = I - Q.$$

Choosing

$$\begin{aligned} a_1(x_0) &= 0, \\ g_y(x_0)f_z(x_0)b_1(x_0) &= \tfrac{1}{2}g_y(x_0)y''(x_0) - g_y(x_0)f_z(x_0)z'(x_0), \\ g_y(x_0)f_z(x_0)k_u(x_0)c_1(x_0) &= -g_y(x_0)f_z(x_0)b_1(x_0) + \tfrac{1}{2}g_y(x_0)f_z(x_0)z''(x_0) \\ &\quad - g_y(x_0)f_z(x_0)k_u(x_0)u'(x_0) \end{aligned}$$

gives consistent initial values for (8) and fixes $Qb_1(x_0)$ as well as $c_1(x_0)$. Now multiplying equation (9) once with P and once with Q (and using $Pk_u(x_0)=0$) determines the remaining coefficients of the expansion $P\beta_0^1=0$, hence $Pb_1(x_0)=0$ and

$$\gamma_1^0 = [g_y(x_0)f_z(x_0)k_u(x_0)]^{-1}g_y(x_0)f_z(x_0)b_1(x_0).$$

The condition (7) for $N=1$ yields a similar index 3 system for $a_2(x)$, $b_2(x)$, and $c_2(x)$ with consistent initial values $a_2(x_0)=0$ and $g_y(x_0)f_z(x_0)b_2(x_0)$, $c_2(x_0)$ suitably chosen. To satisfy the equation

$$k_u(x_0)\gamma_1^1 = -\beta_0^2 + \varepsilon_1 \tag{9'}$$

(where ε_1 depends on known quantities only), we again apply P once and Q once,

$$P\beta_0^2 = P\varepsilon_1, \qquad k_u(x_0)\gamma_1^1 = -Q\beta_0^2 + Q\varepsilon_1 = Qb_2(x_0) + Q\varepsilon_1.$$

These conditions fix $Pb_2(x_0)$ and γ_1^1. Repeating this procedure yields (7) for arbitrary N.

(b) It remains to give the stability estimate. To this aim we will show inductively that the solution of (3) exists and that the differences $\Delta y_{n+1} = y_{n+1} - \hat{y}_{n+1}$, etc. satisfy

$$\Delta y_{n+1} = O(h^{N+1}), \qquad \Delta z_{n+1} = O(h^{N+1}), \qquad \Delta u_{n+1} = O(h^N)$$

for $nh \leq \text{const}$.

By construction one has $\Delta y_0 = 0$ and $\Delta z_0 = 0$. For the general step, one uses inductively

$$y_n = y(x_n) + ha_1(x_n) + O(h^2), \qquad z_n = z(x_n) + O(h)$$

which shows by (8) that $g(y_n + hf(y_n, z_n)) = O(h^2)$.

Thus, Theorem 1 guarantees a unique solution y_{n+1}, z_{n+1}, and u_{n+1} of (3) and, as $u_{n+1} = O(1)$ by Theorem 1, a Lipschitz condition for f and k immediately yields the a priori estimates

$$\Delta y_{n+1} = O(h), \qquad \Delta z_{n+1} = O(h).$$

Now linearizing the difference of (3) and (7) one obtains

$$\Delta y_{n+1} = \Delta y_n + hf_z(\hat{y}_n, \hat{z}_{n+1})\Delta z_{n+1} + O(h\,\|\Delta y_n\| + h^2\,\|\Delta z_{n+1}\|) + O(h^{N+3}), \quad (10a)$$

$$h\Delta z_{n+1} = h\Delta z_n + h^2 k_u(\hat{y}_n, \hat{z}_n)\Delta u_{n+1} + O(h^2\,\|\Delta y_n\| + h^2\,\|\Delta z_n\|) + O(h^{N+3}), \quad (10b)$$

$$0 = g_y(\hat{y}_{n+1})\Delta y_{n+1} + O(h\,\|\Delta y_{n+1}\|) + O(h^{N+2}). \quad (10c)$$

Inserting (10a) and (10b) into (10c) and extracting Δu_{n+1} from the resulting equation gives

$$-h^2(g_y f_z k_u)_n \Delta u_{n+1} = (g_y)_n \Delta y_n + (g_y f_z)_n h\Delta z_n + O(h\,\|\Delta y_n\| + h^2\,\|\Delta z_n\|) + O(h^{N+2}) \quad (11)$$

(with the notation $(g_y f_z)_n = g_y(\hat{y}_n) f_z(\hat{y}_n, \hat{z}_n)$, etc.). Reinserting this into (10b) and (10a) finally leads to the recursion

$$\Delta y_{n+1} = (I - f_z S)_n \Delta y_n + (f_z)_n (I - Sf_z)_n h\Delta z_n + O(h\,\|\Delta y_n\| + h^2\,\|\Delta z_n\|) + (f_z k_u)_n O(h^{N+2}) + O(h^{N+3}), \quad (12a)$$

$$h\Delta z_{n+1} = (I - Sf_z)_n h\Delta z_n - S_n (f_z S)_n \Delta y_n + O(h\,\|\Delta y_n\| + h^2\,\|\Delta z_n\|) + (k_u)_n O(h^{N+2}) + O(h^{N+3}), \quad (12b)$$

where S denotes the matrix $k_u(g_y f_z k_u)^{-1} g_y$. Now, direct application of Lemma 6.5 in Hairer, Lubich, & Roche (1989) gives the desired results

$$\|\Delta y_{n+1}\| = O(h^{N+1}), \qquad \|\Delta z_{n+1}\| = O(h^{N+1}),$$

and together with (10c) and (11) the estimate for the u component $\|\Delta u_{n+1}\| = O(h^N)$. □

4. Convergence of the extrapolation method

This section is devoted to extrapolation based on method (3). We want to investigate the local error as well as the order of convergence. To this aim we shall first define the extrapolation methods.

For a given sequence of positive integers $n_1 < n_2 < n_3 < \cdots$ and a basic stepsize $H > 0$, let $h_j = H/n_j$ denote the corresponding stepsizes and $y_h(x) = y_n$ for $x = x_0 + nh$. The tableau for h extrapolation is then given by

$$T_{j,1} = y_{h_j}(x_0 + H), \qquad T_{j,k+1} = T_{j,k} + \frac{T_{j,k} - T_{j-1,k}}{(n_j/n_{j-k}) - 1}.$$

Each entry $T_{j,k}$ in this tableau can now be regarded as a numerical method for solving equation (1). We shall denote this numerical solution at the gridpoints $x_0 + mH$ by capitals Y_m, Z_m, and U_m, componentwise. The aim of this section is to verify the order table (Table 1).

TABLE 1
Order results for $T_{j,k}$ (if $n_1 \geq 2$)

Component	Local order	Order of convergence
y	k	$\max(1, k-1)$
z	$k-1$	$\max(1, k-1)$
u	$k-1$	$\max(1, k-1)$

4.1 *The Local Error*

The assertion for the local error is proved by Theorem 2. If $n_1 \geqslant 2$, each extrapolation eliminates one power of h in the expansion of the theorem. Thus the local error satisfies

$$T_{j,k} - \begin{pmatrix} y(x_0+H) \\ z(x_0+H) \\ u(x_0+H) \end{pmatrix} = \frac{(-1)^{k+1}}{n_j \cdots n_{j-k+1}} \begin{pmatrix} a_k(x_0+H) \\ b_k(x_0+H) \\ c_k(x_0+H) \end{pmatrix} H^k + O(H^{k+1}).$$

(cf. error formula for polynomial interpolation in Abramowitz & Stegun (1964: Formula 25.2.27)). Since $a_k(x_0) = 0$, the local error of the y component is $O(H^{k+1})$, whereas it is $O(H^k)$ for the other two components.

4.2 *The Global Error*

For $k = 1$ the result is an immediate consequence of Theorem 2. Therefore we have to consider $k \geqslant 2$ only. In this case the global error can be estimated by standard techniques like Lady Windermere's Fan (Hairer, Nørsett, & Wanner, 1987: p. 160). Figure 1 shows this fan for the y component.

To estimate the global errors $Y_N^0 - Y_N^N$ and $Z_N^0 - Z_N^N$ of the y and z components, we analyse the propagation of the local errors, that means we

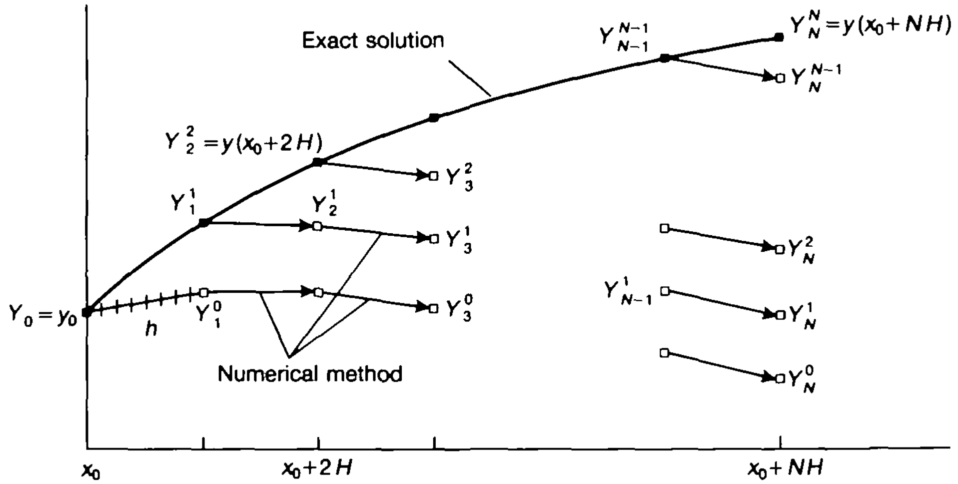

FIG. 1. Lady Windermere's Fan.

estimate the differences $Y_N^{l-1} - Y_N^l$ and $Z_N^{l-1} - Z_N^l$ in terms of the local errors $Y_l^{l-1} - Y_l^l$ and $Z_l^{l-1} - Z_l^l$. We suppose for a moment

$$\|g(Y_m^{l-1} + Hf(Y_m^{l-1}, Z_m^{l-1}))\| \le C_0 H^2, \qquad \|g(Y_m^l + Hf(Y_m^l, Z_m^l))\| \le C_0 H^2, \tag{13}$$

which will be justified inductively below. Because of (13), Theorem 1 guarantees the existence of Y_{m+1}^{l-1}, Y_{m+1}^l, Z_{m+1}^{l-1}, and Z_{m+1}^l. Using now the same techniques as in the proof of Theorem 2, part (b), one can show that the differences $\Delta Y_{m+1}^l = Y_{m+1}^{l-1} - Y_{m+1}^l$ and $\Delta Z_{m+1}^l = Z_{m+1}^{l-1} - Z_{m+1}^l$ satisfy again a recursion like (12)—but with all derivatives now evaluated at the exact solution

$$\begin{aligned} \Delta Y_{m+1}^l &= (I - f_z S)_m \Delta Y_m^l + (f_z)_m (I - Sf_z)_m H \Delta Z_m^l \\ &\quad + O(H \|\Delta Y_m^l\| + H^2 \|\Delta Z_m^l\|), \end{aligned} \tag{14a}$$

$$\begin{aligned} H\Delta Z_{m+1}^l &= (I - Sf_z)_m H \Delta Z_m^l + S_m (f_z S)_m \Delta Y_m^l \\ &\quad + O(H \|\Delta Y_m^l\| + H^2 \|\Delta Z_m^l\|). \end{aligned} \tag{14b}$$

Application of Lemma 6.5 in Hairer, Lubich, & Roche (1989) with $\rho = 0$ thus yields, in the notation of Fig. 1,

$$\|Y_N^{l-1} - Y_N^l\| \le CH^k \quad \text{and} \quad \|Z_N^{l-1} - Z_N^l\| \le CH^k. \tag{15}$$

Summing up from $l = 1$ to N gives the order results for the y and z components as stated in Table 1.

To get the estimate for the u component we start with (10b). Using the notation of Section 3 for the current step at $x_0 + mH$ we obtain

$$h^2 (g_y f_z k_u)_n \Delta u_{n+1} = h(g_y f_z)_n (\Delta z_{n+1} - \Delta z_n) + O(H^{k+2}). \tag{16}$$

To estimate the right-hand side, we multiply (10a) by $(g_y)_{n+1}$ which gives

$$0 = (g_y)_{n+1} \Delta y_n + h(g_y f_z)_n \Delta z_{n+1} + h(g_y f_y)_n \Delta y_n + O(H^{k+2}).$$

Here we used $(g_y)_{n+1} = (g_y)_n + O(h)$ and $(g_y)_{n+1} \Delta y_{n+1} = O(H^{k+2})$. Now multiplying (10a) (for n replaced by $n-1$) by $(g_y)_{n-1}$, one similarly gets

$$(g_y)_{n-1} \Delta y_n = h(g_y f_z)_n \Delta z_n + h(g_y f_y)_n \Delta y_n + O(H^{k+2}).$$

A combination of the last two equations reads

$$h(g_y f_z)_n (\Delta z_{n+1} - \Delta z_n) = -[(g_y)_{n+1} + (g_y)_{n-1}] \Delta y_n + O(H^{k+2}) = O(H^{k+2})$$

which together with (16) implies the desired result

$$U_N^{l-1} - U_N^l = O(H^k). \tag{17}$$

It remains to justify the assumption (13). By Theorem 2 it is satisfied for $m = l$. To treat the general case we use the fact that as long as $U_m - u(x_0 + mH) = O(H)$, the constant C_0 does not affect the recursion (14). Thus, the constants in (15) and (17) can be chosen independently of C_0, which justifies (13) recursively.

5. A numerical example

To show the relevance of our theoretical results to practical computations, we have tested the extrapolation method (3) with the stepnumber sequence

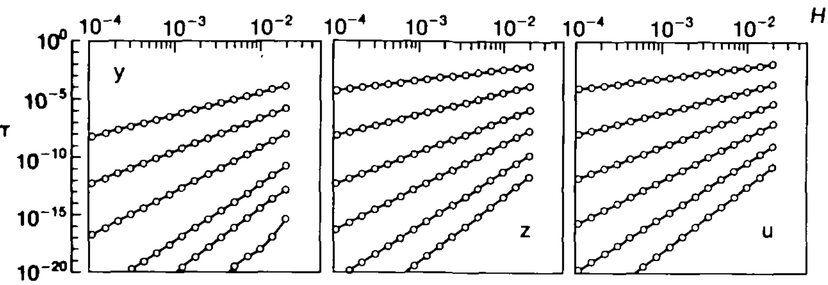

FIG. 2. The local errors (for $j = 1,\dots, 6$).

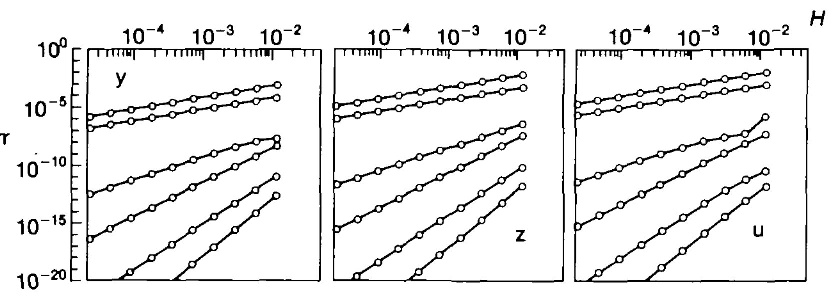

FIG. 3. The global errors at $x = 0{\cdot}1$.

$n_j = \{2, 3, 4, 5, 6, 7,\dots\}$ at the index 3 problem

$$r' = rsv^2, \qquad s' = rsvw, \qquad v' = r^2sv^2u,$$

$$w' = r^2u - v + r^2w^2, \qquad 0 = r^2s - 1,$$

which is of form (1), (2) with $y = (r, s)^T$ and $z = (v, w)^T$.

For the (consistent) initial values $y_0 = (1, 1)^T$ and $z_0 = (1, -2)^T$ the exact solution is given by

$$y(x) = (e^x, e^{-2x})^T, \qquad z(x) = (e^x, -2e^{-2x})^T, \qquad u(x) = e^{-x}.$$

In Fig. 2 the local errors $\|T_{j,j}^{1,2} - y(H)\|$, $\|T_{j,j}^{3,4} - z(H)\|$, and $|T_{j,j}^5 - u(H)|$ are plotted as functions of H. As we used a logarithmic scale, the curves appear as straight lines of slope q whenever the leading term of the error is $O(H^q)$.

The corresponding results for the global error are displayed in Fig. 3. In all cases the obvious order pattern matches the theoretical results perfectly.

Acknowledgements

I would like to thank E. Hairer and Ch. Lubich for their very helpful suggestions. They also gave the impetus for the present study.

This work has been supported by the 'Fonds national suisse de la recherche scientifique'.

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