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A cohomological stability result for projective schemes over surfaces

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Abstract. Let $\pi : X \to X_0$ be a projective morphism of schemes such that X_0 is noetherian and essentially of finite type over a field K. Let $i \in \mathbb{N}_0$, let $\mathscr{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $\mathscr{L}$ be an ample invertible sheaf over X. We show that the set $\mathrm{Ass}_{X_0}\big(\mathscr{R}^i\pi_*(\mathscr{L}^n \otimes_{\mathcal{O}_X} \mathscr{F})\big)$ of associated points of the higher direct image sheaf $\mathscr{R}^i\pi_*(\mathscr{L}^n \otimes_{\mathcal{O}_X} \mathscr{F})$ ultimately becomes constant if n tends to $-\infty$, provided X_0 has dimension $\leqq 2$. If $X_0 = \mathbb{A}^3_K$, this stability result need not hold any more.

To prove this, we show that the set $\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)$ of associated primes of the n-th graded component of the i-th local cohomology module of a finitely generated graded module M over a homogeneous noetherian ring $R = \bigoplus_{n \geqq 0} R_n$ which is essentially of finite type over a field becomes ultimately constant in codimension 2 if n tends to $-\infty$.

1. Introduction

Let $\mathfrak{a}$ be an ideal of the noetherian ring A and let M be a finitely generated A-module. Moreover, let i be a non-negative integer and let $H^i_{\mathfrak{a}}(M)$ be the i-th local cohomology module of M with respect to $\mathfrak{a}$. In 1992, Huneke [11] asked, whether the set $\mathrm{Ass}_A\big(H^i_{\mathfrak{a}}(M)\big)$ of associated primes of the A-module $H^i_{\mathfrak{a}}(M)$ is finite.

If $M = A$ and A is a regular local ring which contains a field K, the previous finiteness question finds an affirmative answer. This was shown by Huneke-Sharp [12] for $\mathrm{char}(K) > 0$ and by Lyubeznik [16] for $\mathrm{char}(K) = 0$. For further and more detailed statements see also [17]. For certain local but non-regular rings, the finiteness of the sets $\mathrm{Ass}_A\big(H^i_{\mathfrak{a}}(A)\big)$ is known, too (cf. [10] or [18], for example). Moreover, without any restriction on A and M, the set $\mathrm{Ass}_A\big(H^i_{\mathfrak{a}}(M)\big)$ is finite, provided the A-modules $H^j_{\mathfrak{a}}(M)$ are finitely generated for all $j < i$ (cf. [6], [13]). This latter result has found various nice extensions (cf. [15] for example).

On the other hand Singh [18] gave the following surprisingly simple example, for which the above finiteness question has a negative answer:

Let $A = \mathbb{Z}[x, y, z, u, v, w]/(xu + yv + zw)$ (with indeterminates x, y, z, u, v, w) and let $\mathfrak{a} := (u, v, w)A$. Then $\mathrm{Ass}_{\mathbb{Z}}(H^3_{\mathfrak{a}}(A))$ is infinite—and hence so is $\mathrm{Ass}_A(H^3_{\mathfrak{a}}(A))$. Katzman [14] gave a similar example of a ring containing a field and later Singh-Swanson [20] developed a method which allows to construct a great variety of such examples.

From the point of view of projective schemes, the above finiteness question finds a natural refinement. Namely, let $R = \bigoplus_{n \geqq 0} R_n$ be a homogeneous noetherian ring, so that the base ring R_0 is noetherian and R is generated over R_0 by finitely many elements of R_1. Moreover, let $R_+ := \bigoplus_{n > 0} R_n$ denote the irrelevant ideal of R, let $i \geqq 0$ and let M be a finitely generated graded R-module. For each $n \in \mathbb{Z}$ let $H^i_{R_+}(M)_n$ be the n-th graded component of the local cohomology module $H^i_{R_+}(M)$ of M with respect to R_+. Then the R_0-module $H^i_{R_+}(M)_n$ is finitely generated for all $n \in \mathbb{Z}$ and vanishes for all $n \gg 0$. Now, instead of our previous finiteness question, we ask:

1.0. *Does the (finite) set* $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ *ultimately become constant, if* $n \to -\infty$?

This question of the *Asymptotic Stability of Associated Primes* obviously plays a crucial rôle in the study of the *Asymptotic Behaviour of Cohomology* (cf. [1]), that is of the R_0-modules $H^i_{R_+}(M)_n$ for $n \ll 0$. It is easy to see, that an affirmative answer to (1.0) yields the finiteness of the set $\mathrm{Ass}_R(H^i_{R_+}(M))$.

On the other hand we may furnish the ring A of Singh with the grading for which x, y, z have degree 0 and u, v, w have degree one. Then $A_0 = \mathbb{Z}[x, y, z]$, and if we choose a prime number p and localize at $(x, y, z, p)A_0$, we get a homogeneous noetherian domain $R = \bigoplus_{n \geqq 0} R_n$ with $R_0 = \mathbb{Z}[x, y, z]_{(p,x,y,z)}$, and such that $\mathrm{Ass}_R(H^3_{R_+}(R))$ is finite, but $\mathrm{Ass}_{R_0}(H^3_{R_+}(R)_n)$ is not asymptotically stable for $n \to -\infty$ (cf. [3], [5]). This shows that (1.0) cannot be answered affirmatively in general, even if the set $\mathrm{Ass}_R(H^i_{R_+}(M))$ is finite. Also, whenever R_0 is a polynomial ring in at least three indeterminates over a field, there is a homogeneous noetherian domain $R = R_0 \oplus R_1 \oplus \cdots$ such that the set $\mathrm{Ass}_{R_0}(H^2_{R_+}(R)_n)$ is not asymtotically stable for $n \to -\infty$ (cf. [1], [2]).

Nevertheless, (1.0) is known to have an affirmative answer in the following cases:

(a) $H^j_{R_+}(M)_n = 0$ for all $j < i$ and all $n \ll 0$ (cf. [4]).

(b) R_0 is essentially of finite type over a field and of dimension $\leqq 1$ (cf. [2]).

In the present paper we show that (1.0) also has an affirmative answer if the base ring R_0 is of dimension 2 and essentially of finite type over a field. Observe that in view of the examples of Singh and Swanson [20] this is the best possible result one may expect, if only a bound on the dimension of the base ring R_0 is imposed.

In fact, we shall prove our result in a more general, geometric setting.

Namely, let X_0 be a noetherian scheme and let $\pi : X \to X_0$ be a projective scheme over X_0. Moreover, let $\mathscr{L}$ be an ample invertible sheaf over X and let $\mathscr{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then, for each $i > 0$ and each $n \in \mathbb{Z}$, the higher direct image sheaf $\mathscr{R}^i\pi_*(\mathscr{L}^n \otimes_{\mathcal{O}_X} \mathscr{F})$ is coherent over $\mathcal{O}_{X_0}$ and vanishes for all $n \gg 0$.

We shall prove the following result on the asymptotic behaviour of the sheaves $\mathscr{R}^i\pi_*(\mathscr{L}^n \otimes_{\mathcal{O}_X} \mathscr{F})$ in which $i \geqq 0$ is fixed and n tends to $-\infty$ (cf. Corollary 5.4):

1.1. *Assume that X_0 is essentially of finite type over a field and of dimension $\leqq 2$. Then, for each $i \geqq 0$ the set*

$$\mathrm{Ass}_{X_0}\big(\mathscr{R}^i\pi_*(\mathscr{L}^n \otimes_{\mathcal{O}_X} \mathscr{F})\big)$$

of points $x_0 \in X_0$ associated to the higher direct image sheaf $\mathscr{R}^i\pi_*(\mathscr{L}^n \otimes_{\mathcal{O}_X} \mathscr{F})$, ultimately becomes constant if $n \to -\infty$.

The restriction on the dimension of the base scheme X_0 may seem surprising, but in accordance with our introductory observations we can say (cf. Example 5.7):

1.2. *If X_0 is an affine* 3*-space over a field, the conclusion of* (1.1) *need not hold any more.*

The above statement (1.1) is a conclusion of more general stability result (cf. Theorem 5.5):

1.3. *Assume that X_0 is essentially of finite type over a field and let $i \in \mathbb{N}_0$.*

Then, the set

$$\mathrm{Ass}_{X_0}\big(\mathscr{R}^i\pi_*(\mathscr{L}^n \otimes \mathscr{F})\big)^{\leqq 2}$$

of points $x_0 \in X_0$ which are associated to $\mathscr{R}^i\pi_*(\mathscr{L}^n \otimes \mathscr{F})$ and of codimension $\leqq 2$ in X_0, ultimately becomes constant if $n \to -\infty$.

This latter result is a consequence of the following asymptotic stability result which holds for affine base schemes (cf. Theorem 5.3):

1.4. *Assume that X_0 is affine and essentially of finite type over a field. Let $i \geqq 0$. Then, the set*

$$\mathrm{Ass}_{\mathcal{O}(X_0)}\big(H^i(X, \mathscr{L}^n \otimes_{\mathcal{O}_X} \mathscr{F})\big)^{\leqq 2}$$

of primes of $\mathcal{O}(X_0)$ which are of height $\leqq 2$ and associated to the cohomology module $H^i(X, \mathscr{L}^n \otimes_{\mathcal{O}_X} \mathscr{F})$, ultimately becomes constant if $n \to -\infty$.

The proof of (1.4) is easy, if once the case of a very ample sheaf $\mathscr{L}$ is established (cf. Proposition 5.2). This latter case follows from an asymptotic stability result on graded components of local cohomology modules.

Namely, let $R = \bigoplus_{n \geqq 0} R_n$ and $R_+ := \bigoplus_{n > 0} R_n$ be as above let $i \geqq 0$ and let M be a finitely generated graded R-module. Then, using our previous notation we have the following stability result (cf. Theorem 4.1):

1.5. *Assume that R_0 is essentially of finite type over a field.*

Then, the set $\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{\leqq 2}$ of primes $\mathfrak{p}_0 \subset R_0$ which are of height $\leqq 2$ and associated to the R_0-module $H^i_{R_+}(M)_n$, ultimately becomes constant if $n \to -\infty$.

We shall prove this result in two steps: Before we establish the general case, we show that the requested asymptotic stability of associated primes holds if R_0 is in addition a domain (cf. Proposition 3.5). This is an essential generalization of a similar but much weaker stability result given in [1].

Our paper is organized as follows: In Section 2 we give a few preliminaries. In Section 3 we prove our main result (1.5) under the hypothesis that R_0 is in addition a domain. In Section 4 we show that this latter hypotheses can be omitted. There we also shall explicitly give an example, essentially due to Singh-Swanson [20], which shows that (1.5) need not hold if R_0 is a three-variate polynomial ring over a field. In Section 5 we apply the results of Section 4 to projective schemes and establish our statements (1.1)–(1.4). For basic notions and notations from Algebraic Geometry and Commutative Algebra we refer to [9] resp. [8].

2. Preliminaries

By $\mathbb{N}$ we denote the set of positive integers, by $\mathbb{N}_0$ the set of non-negative integers.

2.1. Notations and conventions. (A) Throughout this paper let $R = R_0 \oplus R_1 \oplus \cdots$ be a *homogeneous noetherian ring*. So R is $\mathbb{N}_0$-graded, R_0 is noetherian and there are finitely many elements $a_1, \ldots, a_k \in R_1$ such that $R = R_0[a_1, \ldots, a_k]$.

By R_+ we denote the *irrelevant ideal* of R, thus $R_+ = R_1 \oplus R_2 \oplus \cdots$.

Polynomial rings over R_0 are always furnished with their standard grading, so that they are homogeneous.

(B) If $i \in \mathbb{N}_0$ and M is a graded R-module, we write $H^i_{R_+}(M)$ for the *i-th local cohomology module of M with respect to* R_+, and we always furnish this module with its natural grading.

For $n \in \mathbb{Z}$ we denote by $H^i_{R_+}(M)_n$ *the n-th graded component* of $H^i_{R_+}(M)$.

Keep in mind that if the graded R-module M is finitely generated, the R_0-module $H^i_{R_+}(M)_n$ is finitely generated for all $n \in \mathbb{Z}$ and vanishes for all $n \gg 0$.

2.2. Definition. Let $\mathscr{S}$ be a set and let $(\mathscr{S}_n)_{n \in \mathbb{Z}}$ be a family of subsets of $\mathscr{S}$. We say that (the set) $\mathscr{S}_n$ is *asymptotically stable for* $n \to -\infty$ if there is some $n_0 \in \mathbb{Z}$ such that $\mathscr{S}_n = \mathscr{S}_{n_0}$ for all $n \leqq n_0$.

2.3. Notation. (A) Let A be a (commutative unitary) ring. If $\mathfrak{a} \subseteqq A$ is an ideal, we denote the variety $\{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subseteqq \mathfrak{p}\}$ of $\mathfrak{a}$ by $\operatorname{Var}(\mathfrak{a})$. In this situation we write $\min(\mathfrak{a})$ for the set of minimal members of $\operatorname{Var}(\mathfrak{a})$. $\operatorname{Max}(A)$ is used to denote the set of maximal ideals in A, whereas $\operatorname{Min}(A)$ is used to denote the set of minimal primes of A.

(B) If $\mathscr{S} \subseteqq \operatorname{Spec}(A)$ and $\ell \in \mathbb{N}_0$, we write

$$\mathscr{S}^{=\ell} := \{\mathfrak{p} \in \mathscr{S} \mid \operatorname{height}(\mathfrak{p}) = \ell\} \quad \text{and} \quad \mathscr{S}^{\leqq \ell} := \{\mathfrak{p} \in \mathscr{S} \mid \operatorname{height}(\mathfrak{p}) \leqq \ell\}.$$

(C) If T is an A-module we write $\operatorname{Ass}_A(T)$ for the set of associated primes of T in A.

Quite often we shall have to use the following two results on asymptotic stability of sets of associated primes:

2.4. Proposition. *Assume that R_0 is essentially of finite type over a field K (so that $R_0 = S^{-1}R_0'$, where $R_0' = K[a_1, \dots, a_r]$ is a finitely generated K-algebra and $S \subseteqq R_0'$ is multiplicatively closed). Let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module. Then, the set $\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{\leqq 1}$ is asymptotically stable for $n \to -\infty$.*

Proof. See [2], Theorem 3.7. □

2.5. Proposition. *Assume that R_0 is semilocal, essentially of finite type over a field and of dimension $\leqq 2$. Let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module.*

Then, the set $\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)$ is asymptotically stable for $n \to -\infty$.

Proof. See [2], Corollary 4.8. □

In addition, the following two results will be used as technical tools.

2.6. Proposition. *Assume that R_0 is a domain and let M be a finitely generated graded R-module.*

Then, there is an element $x \in R_0 \backslash \{0\}$ such that the $(R_0)_x$-module $H^i_{R_+}(M)_x$ is torsion free (or vanishes) for all $i \in \mathbb{N}_0$.

Proof. See [2], Theorem 2.5. □

2.7. Proposition. *Assume that $(R_0, \mathfrak{m}_0)$ is local and of dimension $\leqq 1$.*

Then, for each $i \in \mathbb{N}_0$ and each finitely generated graded R-module M, the R-module $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M)\big)$ is artinian.

Proof. See [3], Theorem 2.5 b). □

3. Integral base rings

3.1. Lemma. *Let $r \in \mathbb{N}$, let $(A, \mathfrak{m})$ be a noetherian local ring and let $x \in \mathfrak{m}$. Let T be a finitely generated A-module such that $\dim_A(T) \leqq 2$ and $\dim_A(T/xT) \leqq 1$. Assume that $x^r\Gamma_{\mathfrak{p}A_\mathfrak{p}}(T_\mathfrak{p}) = 0$ for each $\mathfrak{p} \in \mathrm{Ass}_A(T)$ with $\dim(A/\mathfrak{p}) = 1$. Assume in addition, that $\mathfrak{m} \in \mathrm{Ass}_A(T)$.*

Then $\mathfrak{m} \in \mathrm{Ass}_A(T/x^{r+1}T)$.

Proof. It suffices to show that $\Gamma_\mathfrak{m}(T) \nsubseteq x^{r+1}T$. As $\mathfrak{m} \in \mathrm{Ass}_A(T)$ we have $\Gamma_\mathfrak{m}(T) \neq 0$. Set

$$\mathscr{S} := \{\mathfrak{p} \in \mathrm{Ass}_A(T) \mid \dim(A/\mathfrak{p}) = 1\}.$$

Assume first, that $\mathscr{S} = \emptyset$. As $\dim_A(T/xT) \leqq 1$ and $\dim(T) \leqq 2$ it follows that x avoids all members of $\mathrm{Ass}_A(T) \backslash \{\mathfrak{m}\}$, so that x is a non-zero divisor with respect to $T/\Gamma_{\mathfrak{m}}(T)$. We obtain $xT \cap \Gamma_{\mathfrak{m}}(T) = x(\Gamma_{\mathfrak{m}}(T) \underset{T}{:} x) = x\Gamma_{\mathfrak{m}}(T)$, hence by Nakayama $xT \cap \Gamma_{\mathfrak{m}}(T) \subsetneqq \Gamma_{\mathfrak{m}}(T)$, thus $\Gamma_{\mathfrak{m}}(T) \nsubseteq xT$. Therefore $\Gamma_{\mathfrak{m}}(T) \nsubseteq x^{r+1}T$.

So, let $\mathscr{S} \neq \emptyset$. Choose $\mathfrak{p} \in \mathscr{S}$. Then $0 : T \subseteqq \mathfrak{p}$ and $\Gamma_{\mathfrak{p}A_{\mathfrak{p}}}(T_{\mathfrak{p}}) \neq 0$. As $x^r\Gamma_{\mathfrak{p}A_{\mathfrak{p}}}(T_{\mathfrak{p}}) = 0$ it follows $x \in \mathfrak{p}$, thus $\mathfrak{p} \in \operatorname{Var}(xA + (0 \underset{A}{\overset{A}{:}} T))\backslash\{\mathfrak{m}\} = \operatorname{Supp}_A(T/xT)\backslash\{\mathfrak{m}\}$. Therefore $\mathscr{S} \subseteqq \operatorname{Supp}_A(T/xT)\backslash\{\mathfrak{m}\}$.

Now, let $\mathfrak{p} \in \operatorname{Supp}_A(T/xT)\backslash\{\mathfrak{m}\}$. As $\dim_A(T/xT) \leqq 1$ it follows that $\mathfrak{p}$ is generic in $\operatorname{Supp}_A(T/xT) = \operatorname{Var}(xA + (0 \underset{A}{:} T))$. Therefore $xA_{\mathfrak{p}} + (0 \underset{A\mathfrak{p}}{:} T_{\mathfrak{p}}) = (xA + (0 \underset{A}{:} T))_{\mathfrak{p}}$ is a $\mathfrak{p}A_{\mathfrak{p}}$-primary ideal of $A_{\mathfrak{p}}$, hence $\Gamma_{xA}(T)_{\mathfrak{p}} \cong \Gamma_{xA_{\mathfrak{p}}}(T_{\mathfrak{p}}) = \Gamma_{(xA_{\mathfrak{p}}+(0 \underset{A_{\mathfrak{p}}}{:} T_{\mathfrak{p}}))}(T_{\mathfrak{p}}) = \Gamma_{\mathfrak{p}A_{\mathfrak{p}}}(T_{\mathfrak{p}})$.

Assume first that $\mathfrak{p} \notin \operatorname{Ass}_R(T)$. Then $\Gamma_{\mathfrak{p}A_{\mathfrak{p}}}(T_{\mathfrak{p}}) = 0$ and hence $x^r\Gamma_{\mathfrak{p}A_{\mathfrak{p}}}(T_{\mathfrak{p}}) = 0$. If $\mathfrak{p} \in \operatorname{Ass}_R(T)$, then $\mathfrak{p} \in \mathscr{S}$, hence $x^r\Gamma_{\mathfrak{p}A_{\mathfrak{p}}}(T_{\mathfrak{p}}) = 0$. It follows that $x^r\Gamma_{xA}(T)_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Supp}_A(T/xT)\backslash\{\mathfrak{m}\}$, hence $\operatorname{Supp}_A(x^r\Gamma_{xA}(T)) \subseteqq \{\mathfrak{m}\}$. So, there is some $n \in \mathbb{N}$ with $\mathfrak{m}^n x^r \Gamma_{xA}(T) = 0$.

Assume now, that $\Gamma_{\mathfrak{m}}(T) \subseteqq x^{r+1}T$. Let $t \in \Gamma_{\mathfrak{m}}(T)$. Then, there is some $s \in T$ with $t = x^{r+1}s$. Moreover $\Gamma_{\mathfrak{m}}(T) \subseteqq \Gamma_{xA}(T)$ shows that $x^{r+1}s = t \in \Gamma_{xA}(T)$, so that $s \in \Gamma_{xA}(T)$. Consequently $\mathfrak{m}^n x^r s = 0$, whence $x^r s \in \Gamma_{\mathfrak{m}}(T)$, thus $t = x(x^r s) \in x\Gamma_{\mathfrak{m}}(T)$. This yields $\Gamma_{\mathfrak{m}}(T) \subseteqq x\Gamma_{\mathfrak{m}}(T)$ and hence the contradiction that $\Gamma_{\mathfrak{m}}(T) = 0$. □

3.2. Lemma. *Let $m \in \mathbb{Z}$, let R_0 be essentially of finite type over a field, let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module. Assume that*

$$\# \bigcup_{n \leqq m} \operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{=2} < \infty.$$

Then $\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leqq 2}$ is asymptotically stable for $n \to -\infty$.

Proof. Observe that

$$\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leqq 2} = \operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leqq 1} \mathbin{\dot\cup} \operatorname{Ass}_{R_+}(H^i_{R_+}(M)_n)^{=2} \quad \text{for all } n \in \mathbb{Z}.$$

According to Proposition 2.4 the set $\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leqq 1}$ is asymptotically stable for $n \to -\infty$. Therefore, the set $\mathscr{S} := \bigcup_{n \leqq m} \operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leqq 2}$ is finite.

According to Proposition 2.5 the set $\mathscr{T}_n^{(\mathfrak{p}_0)} := \operatorname{Ass}_{(R_0)_{\mathfrak{p}_0}}(H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n)$ is asymptotically stable for $n \to -\infty$ and all $\mathfrak{p}_0 \in \operatorname{Spec}(R_0)^{\leqq 2}$. As $H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n \cong (H^i_{R_+}(M)_n)_{\mathfrak{p}_0}$, we have $\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n) = \bigcup_{\mathfrak{p}_0 \in \mathscr{S}} \{\mathfrak{q}_0 \cap R_0 \mid \mathfrak{q}_0 \in \mathscr{T}_n^{(\mathfrak{p}_0)}\}$ and this gives our claim. □

3.3. Lemma. *Let $B = \bigoplus_{n \geqq 0} B_n$ be a positively graded ring such that $(B_0, \mathfrak{n}_0)$ is noetherian local and let $G = \bigoplus_{n \in \mathbb{Z}} G_n$ be a graded B-module which is artinian and such that the B_0-module G_n is finitely generated for all $n \in \mathbb{Z}$.*

Then, there is some $r \in \mathbb{N}$ such that $\mathfrak{n}_0^r G = 0$.

Proof. For each $n \in \mathbb{Z}$, the B_0-module $G_n \cong G_{\geqq n}/G_{\geqq(n+1)}$ is artinian and finitely generated. As G is artinian, there is some $r \in \mathbb{N}$ with $\mathfrak{n}_0^r G = \mathfrak{n}_0^{r+1} G$. So, for each $n \in \mathbb{Z}$ we have $\mathfrak{n}_0^r(G_n) = (\mathfrak{n}_0^r G)_n = (\mathfrak{n}_0^{r+1} G)_n = \mathfrak{n}_0^{r+1}(G_n) = \mathfrak{n}_0(\mathfrak{n}_0^r(G_n))$, thus $\mathfrak{n}_0^r G_n = 0$ by Nakayama. □

3.4. Lemma. *Let $k, \ell, m \in \mathbb{Z}$ be such that $\ell \geqq 0$ and $k > 0$. Let $x_1, \dots, x_k$ be indeterminates and assume that there is a surjective homomorphism of graded rings*

$$R_0[\underline{x}] := R_0[x_1, \dots, x_k] \twoheadrightarrow R.$$

Assume that

$$\# \bigcup_{n \leqq m} \mathrm{Ass}_{R_0}\big(H^k_{R_0[\underline{x}]_+}(N)_n\big)^{\leqq \ell} < \infty$$

for all finitely generated graded $R_0[\underline{x}]$-modules N.

Then, for all finitely generated graded R-modules M and all $i \in \mathbb{N}_0$ we have

$$\# \bigcup_{n \in \mathbb{Z}} \mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{\leqq \ell} < \infty.$$

Proof. By the graded base ring independence property of local cohomology we may assume that $R = R_0[\underline{x}]$. For $i > k$ we have $H^i_{R_+}(M) = 0$ and there is nothing to show. For $i = k$ our claim follows as $H^i_{R_+}(M)_n = 0$ for all $n \gg 0$. So, let $i < k$. There is an exact sequence of finitely generated graded R-modules

$$0 \to N \to F \to M \to 0$$

in which $F = \bigoplus_{j=1}^{r} R(a_j)$ is a graded free R-module with $a_j \in \mathbb{Z}$ for $j = 1, \dots, r$. As $R = R_0[\underline{x}]$ we have $H^i_{R_+}(F) = 0$, so that for each $n \in \mathbb{Z}$ there is a monomorphism of R_0-modules $0 \to H^i_{R_+}(M)_n \to H^{i+1}_{R_+}(N)_n$ and hence $\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{\leqq \ell} \subseteqq \mathrm{Ass}_{R_0}\big(H^{i+1}_{R_+}(N)_n\big)^{\leqq \ell}$. Now, we may conclude by descending induction on i. □

3.5. Proposition. *Let R_0 be a domain which is essentially of finite type over a field, let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module.*

Then $\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{\leqq 2}$ is asymptotically stable for $n \to -\infty$.

Proof. According to Lemma 3.2 it suffices to show that the set

$$\bigcup_{n \leqq m} \mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{=2} =: \mathscr{W}^i_m$$

is finite for some $m \in \mathbb{Z}$. To prove this, we use Lemma 3.4 and restrict ourselves to the case where $R = R_0[\underline{x}] = R_0[x_1, \dots, x_k]$ is a polynomial ring and where $i = k$. According to Proposition 2.6 there is an element $x \in R_0 \setminus \{0\}$ such that the $(R_0)_x$-module $H^k_{R_+}(M)_x$ is torsion-free (or vanishes). Therefore

$$\mathrm{Ass}_{R_0}\big(H^k_{R_+}(M)_n\big) \setminus \{0\} \subseteqq \mathrm{Var}(xR_0), \quad \forall n \in \mathbb{Z}.$$

According to Proposition 2.4 the set $\mathrm{Ass}_{R_0}\big(H^k_{R_+}(M)_n\big)^{\leqq 1}$ is asymptotically stable for $n \to -\infty$. So, there is an integer m and a finite set $U \subseteqq \mathrm{Var}(xR_0)$ such that

$$\mathrm{Ass}_{R_0}\big(H^k_{R_+}(M)_n\big)^{=1} = U, \quad \forall n \leqq m.$$

Now, let $\mathfrak{p}_0 \in U$. Then the graded $R_{\mathfrak{p}_0}$-module $\Gamma_{\mathfrak{p}_0 R}\big(H^k_{R_+}(M)\big)_{\mathfrak{p}_0} \cong \Gamma_{\mathfrak{p}_0 R_{\mathfrak{p}_0}}\big(H^k_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})\big)$ is artinian by Proposition 2.7. Moreover, for each $n \in \mathbb{Z}$ the n-th graded component $\Gamma_{\mathfrak{p}_0}\big(H^k_{R_+}(M)_n\big)_{\mathfrak{p}_0}$ of this module is finitely generated over $(R_{\mathfrak{p}_0})_0 = (R_0)_{\mathfrak{p}_0}$. As $x \in \mathfrak{p}_0$ we thus find a positive integer $r(\mathfrak{p}_0)$ such that $x^{r(\mathfrak{p}_0)}\Gamma_{\mathfrak{p}_0}\big(H^k_{R_+}(M)\big)_{\mathfrak{p}_0} = 0$ (cf. Lemma 3.3). Thus

$$x^{r(\mathfrak{p}_0)}\Gamma_{\mathfrak{p}_0}\big(H^k_{R_+}(M)_n\big)_{\mathfrak{p}_0} = 0, \quad \forall n \in \mathbb{Z}.$$

Now, set $r := 1$ if $U = \emptyset$, and $r := \max\{r(\mathfrak{p}_0) \mid \mathfrak{p}_0 \in U\}$ otherwise.

By the base ring independence of local cohomology we have isomorphisms of R_0-modules

$$H^k_{R_+}(M/x^{r+1}M)_n \cong H^k_{(R/x^{r+1}R)_+}(M/x^{r+1}M)_n$$

which yield that

$$\begin{aligned}&\mathrm{Ass}_{R_0}\big(H^k_{R_+}(M/x^{r+1}M)_n\big)^{\leqq 2}\\ &\quad = \big\{\mathfrak{p}_0 \in \mathrm{Var}(xR_0) \mid \mathfrak{p}_0/x^{r+1}R_0 \in \mathrm{Ass}_{R_0/x^{r+1}R_0}\big(H^k_{(R/x^{r+1}R)_+}(M/x^{r+1}M)_n\big)^{\leqq 1}\big\}.\end{aligned}$$

As the set $\mathrm{Ass}_{R_0/x^{r+1}R_0}\big(H^k_{(R/x^{r+1}R)_+}(M/x^{r+1}M)_n\big)^{\leqq 1}$ is asymptotically stable for $n \to -\infty$ (cf. Proposition 2.4), it follows that the set

$$\mathscr{T} := \bigcup_{n \in \mathbb{Z}} \mathrm{Ass}_{R_0}\big(H^k_{R_+}(M/x^{r+1}M)_n\big)^{\leqq 2}$$

is finite.

Now, let $n \leqq m$ and let $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}\big(H^k_{R_+}(M)_n\big)^{=2}$. Then $(A, \mathfrak{m}) := \big((R_0)_{\mathfrak{p}_0}, \mathfrak{p}_0(R_0)_{\mathfrak{p}_0}\big)$ is a noetherian local domain of dimension 2 with $x \in \mathfrak{m}\setminus\{0\}$ and $\mathfrak{m}$ is associated to $T := \big(H^k_{R_+}(M)_n\big)_{\mathfrak{p}_0}$. Moreover, $\dim_A(T/xT) \leqq \dim(A/xA) = 1$. In addition, if

$$\mathfrak{p} \in \mathrm{Ass}_A(T) \quad \text{with } \dim(A/\mathfrak{p}) = 1,$$

we may write $\mathfrak{p} = \mathfrak{q}_0 A$ with $\mathfrak{q}_0 := \mathfrak{p} \cap R_0 \in \mathrm{Ass}_{R_0}\big(H^k_{R_+}(M)_n\big)^{=1} = U$. Therefore

$$x^r\Gamma_{\mathfrak{p}A_{\mathfrak{p}}}(T_{\mathfrak{p}}) = x^r\Gamma_{\mathfrak{q}_0(R_0)_{\mathfrak{q}_0}}\big(H^k_{R_+}(M)_n\big)_{\mathfrak{q}_0} = 0.$$

According to Lemma 3.1 we thus get $\mathfrak{m} \in \mathrm{Ass}_A(T/x^{r+1}T)$, whence

$$\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}\big(H^k_{R_+}(M)_n/x^{r+1}H^k_{R_+}(M)_n\big)^{=2}.$$

Now, we consider the following commutative diagram of graded R-modules with exact first row in which π is the canonical epimorphism:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M/(0 \underset{M}{:} x^{r+1}) & \longrightarrow & M & \longrightarrow & M/x^{r+1}M & \longrightarrow & 0 \\ & & \pi \Big\uparrow & & \Big\| & & & & \\ & & M & \xrightarrow{x^{r+1}} & M. & & & & \end{array}$$

As $H^{k+1}_{R_+}(\mathrm{Ker}(\pi)) = 0$ we thus get a commutative diagram of R_0-modules with exact first row and an epimorphism $\psi := H^k_{R_+}(\pi)$:

$$\begin{array}{ccccc} H^k_{R_+}\big(M/(0 \underset{M}{:} x^{r+1})\big)_n & \xrightarrow{\varphi} & H^k_{R_+}(M)_n & \longrightarrow & H^k_{R_+}(M/x^{r+1}M)_n \\ \psi \Big\uparrow & & \Big\| & & \\ H^k_{R_+}(M)_n & \xrightarrow{x^{r+1}} & H^k_{R_+}(M)_n. & & \end{array}$$

It follows that $\mathrm{Im}(\varphi) = x^{r+1}H^k_{R_+}(M)_n$, so that there is a monomorphism

$$0 \to H^k_{R_+}(M)_n/x^{r+1}H^k_{R_+}(M)_n \to H^k_{R_+}(M/x^{r+1}M)_n.$$

Therefore $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}\big(H^k_{R_+}(M/x^{r+1}M)_n\big)^{=2} \subseteqq \mathscr{T}$. So, $\mathscr{W}^k_m \subseteqq \mathscr{T}$. As $\mathscr{T}$ is finite, this proves our claim. □

4. General base rings

4.1. Theorem. *Let R_0 be essentially of finite type over a field. Let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module.*

Then, the set $\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{\leqq 2}$ is asymptotically stable for $n \to -\infty$.

Proof. By the graded base ring independence property of local cohomology we may assume that $R = R_0[\underline{x}] = R_0[x_1, \ldots, x_k]$ is a polynomial ring. Moreover, we may write $R_0 = S_0^{-1}A_0$, where A_0 is a subring of R_0 which is of finite type over a field K, and where $S_0 \subseteqq A_0$ is multiplicatively closed. Let $m_1, \ldots, m_s \in M$ be homogeneous elements such that $M = \sum_{j=1}^{s} Rm_i$. As $A_0[\underline{x}]$ is a subring of $R_0[\underline{x}]$ we may consider the finitely generated graded $A_0[\underline{x}]$-module $N := \sum_{j=1}^{s} A_0[\underline{x}]m_j$. As $R = S_0^{-1}A_0[\underline{x}]$ and $M = S_0^{-1}N$, the graded flat base change property of local cohomology yields an isomorphism of R_0-modules

$$H^i_{R_+}(M)_n = H^i_{(S_0^{-1}A_0[\underline{x}])_+}(S_0^{-1}N)_n \cong S_0^{-1}H^i_{A_0[\underline{x}]_+}(N)_n$$

for each $n \in \mathbb{Z}$. So, for each $n \in \mathbb{Z}$ we have

$$\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{\leqq 2} = \big\{\mathfrak{q}_0 R_0 \mid \mathfrak{q}_0 \in \mathrm{Ass}_{A_0}\big(H^i_{A_0[\underline{x}]_+}(N)_n\big)^{\leqq 2} \text{ and } \mathfrak{q}_0 \cap S_0 = \emptyset\big\}.$$

Thus it suffices to show that $\mathrm{Ass}_{A_0}\big(H^i_{A_0[\underline{x}]_+}(N)_n\big)^{\leqq 2}$ is asymptotically stable for $n \to -\infty$. Therefore we may assume that R_0 is of finite type over the field K.

We do this by induction on the dimension deficiency

$$\delta = \delta(R_0) := \dim(R_0) - \min\{\dim(R_0/\mathfrak{q}_0) \mid \mathfrak{q}_0 \in \mathrm{Min}(R_0)\}.$$

Assume first that $\delta = 0$, so that $\dim(R_0/\mathfrak{q}_0) = \dim(R_0)$ for each $\mathfrak{q}_0 \in \mathrm{Min}(R_0)$. Let $d := \dim(R_0)$. By the Normalization Lemma there is a polynomial ring $B_0 := K[y_1, \ldots, y_d] \subseteqq R_0$ such that R_0 is a finite integral extension of B_0. Moreover $\mathfrak{q}_0 \cap B_0 = 0$ for each $\mathfrak{q}_0 \in \mathrm{Min}(R_0)$ so that

$$\mathrm{height}(\mathfrak{p}_0 \cap B_0) = \mathrm{height}(\mathfrak{p}_0) \quad \text{for all } \mathfrak{p}_0 \in \mathrm{Spec}(R_0).$$

As $R = R_0[\underline{x}]$ is a finite integral extension of $B := B_0[\underline{x}]$, we see that M is a finitely generated graded B-module. So, according to Proposition 3.5 the set $\mathrm{Ass}_{B_0}\big(H^i_{B_+}(M)_n\big)^{\leqq 2}$ is asymptotically stable for $n \to -\infty$. In particular, the set

$$\mathscr{W}' := \bigcup_{n \in \mathbb{Z}} \mathrm{Ass}_{B_0}\big(H^i_{B_+}(M)_n\big)^{=2}$$

is finite. As R_0 is a finite integral extension of B_0, it follows that the set

$$\mathscr{W} := \{\mathfrak{p}_0 \in \mathrm{Spec}(R_0) \mid \mathfrak{p}_0 \cap B_0 \in \mathscr{W}'\}$$

is finite.

Now, let $n \in \mathbb{Z}$ and let $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{=2}$. By the graded base ring independence property of local cohomology there is an isomorphism of B_0-modules $H^i_{R_+}(M)_n \cong H^i_{B_+}(M)_n$. Therefore $\mathfrak{p}_0 \cap B_0 \in \mathrm{Ass}_{B_0}\big(H^i_{B_+}(M)_n\big)$. As

$$\mathrm{height}(\mathfrak{p}_0 \cap B_0) = \mathrm{height}(\mathfrak{p}_0) = 2$$

it follows that $\mathfrak{p}_0 \in \mathscr{W}$. So, the set $\bigcup_{n \in \mathbb{Z}} \mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)\big)^{=2}$ is finite and hence by Lemma 3.2 the set $\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{\leqq 2}$ is asymptotically stable for $n \to -\infty$.

Next, let $\delta = \delta(R_0) > 0$. We write $\mathfrak{q}_0^{(1)}, \mathfrak{q}_0^{(2)}, \ldots, \mathfrak{q}_0^{(r)}$ for the different minimal primes of R_0, assuming that $\dim(R_0/\mathfrak{q}_0^{(j)}) = \dim(R_0)$ for $j = 1, \ldots, s$ and $\dim(R_0/\mathfrak{q}_0^{(\ell)}) < \dim(R_0)$ for $\ell = s+1, \ldots, r$ for some $s \in \{1, \ldots, r-1\}$. By prime avoidance we find elements

$$f_1, \ldots, f_p \in \bigcap_{j=1}^{s} \mathfrak{q}_0^{(j)} \setminus \bigcup_{\ell=s+1}^{r} \mathfrak{q}_0^{(\ell)},$$

$$g_1, \ldots, g_q \in \bigcap_{\ell=s+1}^{r} \mathfrak{q}_0^{(\ell)} \setminus \bigcup_{j=1}^{s} \mathfrak{q}_0^{(j)}$$

such that

$$\mathfrak{a}_0 := \bigcap_{j=1}^{s} \mathfrak{q}_0^{(j)} = (f_1, \ldots, f_p) \quad \text{and} \quad \mathfrak{b}_0 := \bigcap_{\ell=s+1}^{r} \mathfrak{q}_0^{(\ell)} = (g_1, \ldots, g_q).$$

It follows

$$\bigcup_{\mu=1}^{p} \mathrm{Spec}(R_0)_{f_\mu} \cup \bigcup_{\nu=1}^{q} \mathrm{Spec}(R_0)_{f_\nu} = \mathrm{Spec}(R_0) \setminus \mathrm{Var}(\mathfrak{a}_0 + \mathfrak{b}_0).$$

Let $\mathfrak{p}_0 \in \mathrm{Var}(\mathfrak{a}_0 + \mathfrak{b}_0)$. Then, there are indices $j \in \{1, \dots, s\}$ and $\ell \in \{s+1, \dots, r\}$ such that $\mathfrak{q}_0^{(j)} \subseteqq \mathfrak{p}_0$ and $\mathfrak{q}_0^{(\ell)} \subseteqq \mathfrak{p}_0$. As R_0 is of finite type over a field, it follows

$$\begin{aligned}\mathrm{height}(\mathfrak{p}_0) = \mathrm{height}(\mathfrak{p}_0/\mathfrak{q}_0^{(j)}) &= \dim(R_0/\mathfrak{q}_0^{(j)}) - \dim(R_0/\mathfrak{p}_0) > \dim(R_0/\mathfrak{q}_0^{(\ell)}) - \dim(R_0/\mathfrak{p}_0)\\ &= \mathrm{height}(\mathfrak{p}_0/\mathfrak{q}_0^{(\ell)}) > 0,\end{aligned}$$

so that $\mathrm{height}(\mathfrak{p}_0) \geqq 2$. This shows that the set $\mathrm{Var}(\mathfrak{a}_0 + \mathfrak{b}_0)^{=2}$ consists of minimal primes of $\mathfrak{a}_0 + \mathfrak{b}_0$ and hence is finite. Consequently the set

$$\mathscr{S} := \mathrm{Spec}(R_0)^{=2} \Big\backslash \left[\bigcup_{\mu=1}^{p} \mathrm{Spec}(R_0)_{f\mu} \cup \bigcup_{v=1}^{q} \mathrm{Spec}(R_0)_{fv}\right]$$

is finite.

Now, observe that

$$\begin{aligned}\mathrm{Min}\big((R_0)_{f_\mu}\big) &= \{(\mathfrak{q}_0^{(\ell)})_{f_\mu} \mid \ell = s+1, \dots, r\} \quad (\mu = 1, \dots, p),\\ \mathrm{Min}\big((R_0)_{g_v}\big) &= \{(\mathfrak{q}_0^{(j)})_{g_v} \mid j = 1, \dots, s\} \qquad (v = 1, \dots, q).\end{aligned}$$

Using again that R_0 is of finite type over a field we thus get

$$\begin{aligned}\delta\big((R_0)_{f_\mu}\big) &= \max_{\ell=s+1}^{r}\{\dim(R_0/\mathfrak{q}_0^{(\ell)})\} - \min_{\ell=s+1}^{r}\{\dim(R_0/\mathfrak{q}_0^{(\ell)})\}\\ &< \dim(R_0) - \min_{t=1}^{r}\{\dim(R/\mathfrak{q}_0^{(t)})\} = \delta(R_0)\end{aligned}$$

for $\mu = 1, \dots, p$. Moreover it follows that $\delta\big((R_0)_{g_v}\big) = 0$ for $v = 1, \dots, q$.

So by induction, for all $\mu \in \{1, \dots, p\}$ and all $v \in \{1, \dots, q\}$, the sets

$$\mathrm{Ass}_{(R_0)_{f_\mu}}\big(H^i_{(R_{f_\mu})_+}(M_{f_\mu})_n\big)^{\leqq 2} \quad \text{and} \quad \mathrm{Ass}_{(R_0)_{g_v}}\big(H^i_{(R_{g_v})_+}(M_{g_v})_n\big)^{\leqq 2}$$

are asymptotically stable for $n \to -\infty$. By the graded flat base change property of local cohomology it follows easily that the sets

$$\begin{aligned}\mathscr{W}_\mu &:= \bigcup_{n \in \mathbb{Z}} \big[\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{=2} \cap \mathrm{Spec}(R_0)_{f_\mu}\big],\\ \mathscr{V}_v &:= \bigcup_{n \in \mathbb{Z}} \big[\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{=2} \cap \mathrm{Spec}(R_0)_{g_v}\big]\end{aligned}$$

are finite for all $\mu \in \{1, \dots, p\}$ and all $v \in \{1, \dots, q\}$.

As $\bigcup_{n \in \mathbb{Z}} \mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)^{=2} \subseteqq \bigcup_{\mu=1}^{p} \mathscr{W}_\mu \cup \bigcup_{v=1}^{q} \mathscr{V}_v \cup \mathscr{S}$ our claim follows by Lemma 3.2. □

4.2. Corollary. *Let R_0, i and M as in Theorem* 4.1, *and let* $\dim(R_0) \leqq 2$.

Then, the set $\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)$ *is asymptotically stable for* $n \to -\infty$.

4.3. Example (cf. [20], Remark 4.2, [1], Example (7.5)). Let K be a field and let x, y, z, u, v be indeterminates. Let $R_0 = K[x, y, z]$ and consider the homogeneous noetherian R_0-algebra

$$R := R_0[u, v]/(y^2u^2 + xyzuv + z^2v^2).$$

Then, the set $\bigcup_{n \leqq 0} \mathrm{Ass}_{R_0}\big(H^2_{R_+}(R)_n\big)$ is infinite, so that the set $\mathrm{Ass}_{R_0}\big(H^2_{R_+}(R)_n\big)$ is not asymptotically stable for $n \to -\infty$.

This shows, that the conclusion of Corollary 4.2 need not hold if $\dim(R_0) \geqq 3$.

5. Applications to ample divisors

5.1. Notations and conventions. (A) Throughout this section let X_0 denote a noetherian scheme and let $\pi : X \to X_0$ be a projective scheme over X with very ample sheaf $\mathcal{O}_X(1)$.

(B) Let $\ell \in \mathbb{N}_0$. If Y is a noetherian scheme and if $Z \subseteqq Y$, we set

$$Z^{\leqq \ell} := \{z \in Z \mid \mathrm{codim}_Y(z) \leqq \ell\},$$

where $\mathrm{codim}_Y(z) = \dim(\mathcal{O}_{Y,z})$ denotes the codimension of the (closure of the) point $z \in Y$.

(C) If Y is a scheme and $\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$-modules, we write $\mathrm{Ass}_Y(\mathcal{F})$ for the set $\{y \in Y \mid \mathfrak{m}_{Y,y} \in \mathrm{Ass}_{\mathcal{O}_{Y,y}}(\mathcal{F}_y)\}$ of points in Y which are associated to $\mathcal{F}$.

(D) The symbol $\tilde{\ }$ is used to denote induced sheaves.

5.2. Proposition. *Assume that X_0 is affine and essentially of finite type over a field. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $i \in \mathbb{N}_0$.*

Then, the set $\mathrm{Ass}_{\mathcal{O}(X_0)}\big(H^i\big(X, \mathcal{F}(n)\big)\big)^{\leqq 2}$ *is asymptotically stable for* $n \to -\infty$.

Proof. Let $R_0 := \mathcal{O}(X_0)$. Then there is a homogeneous noetherian R_0-algebra $R = \bigoplus_{n \geqq 0} R_n$ such that $X_0 = \mathrm{Spec}(R_0)$, $X = \mathrm{Proj}(R)$ and $\mathcal{O}_X(1) = R(1)^{\sim}$. Moreover there is a finitely generated graded R-module M such that $\mathcal{F} = \tilde{M}$. For each $n \in \mathbb{Z}$ the Serre-Grothendieck correspondence yields a short exact sequence of R_0-modules

$$0 \to H^0_{R_+}(M)_n \to M_n \to H^0\big(X, \mathcal{F}(n)\big) \to H^1_{R_+}(M)_n \to 0$$

and isomorphisms of R_0-modules

$$H^j\big(X, \mathcal{F}(n)\big) \cong H^{j+1}_{R_+}(M)_n \quad \text{for all } j > 0.$$

Now, we conclude by Theorem 4.1. □

5.3. Theorem. *Assume that X_0 is affine and essentially of finite type over a field. Let $\mathscr{F}$ be a coherent sheaf of $\mathscr{O}_X$-modules, let $\mathscr{L}$ be an ample invertible sheaf of $\mathscr{O}_X$-modules and let $i \in \mathbb{N}_0$.*

Then, the set $\mathrm{Ass}_{\mathscr{O}(X_0)}\big(H^i(X, \mathscr{L}^n \otimes_{\mathscr{O}_X} \mathscr{F})\big)^{\leqq 2}$ *is asymptotically stable for $n \to -\infty$.*

Proof. There is an integer $r_0 \in \mathbb{N}$ such that $\mathscr{L}^r$ is very ample for each $r > r_0$. Fix such an r and let $t \in \{0, \dots, r-1\}$. If we apply Proposition 5.2 with $\mathscr{L}^r$ instead of $\mathscr{O}_X(1)$ and $\mathscr{L}^t \otimes_{\mathscr{O}_X} \mathscr{F}$ instead of $\mathscr{F}$ we find an integer n_t and a set $\mathscr{S}_t \subseteqq \mathrm{Spec}\big(\mathscr{O}(X_0)\big)$ such that

$$\mathrm{Ass}_{\mathscr{O}(X_0)}\big(H^i(X, \mathscr{L}^{nr+t} \otimes_{\mathscr{O}_X} \mathscr{F})\big)^{\leqq 2} = \mathscr{S}_t \quad \text{for all } n \leqq n_t.$$

Choosing a second integer $s > r_0$ we find some $m \in \mathbb{Z}$ and some set $\mathscr{T} \subseteqq \mathrm{Spec}\big(\mathscr{O}(X_0)\big)$ such that

$$\mathrm{Ass}_{\mathscr{O}(X_0)}\big(H^i(X, \mathscr{L}^{ns} \otimes \mathscr{F})\big)^{\leqq 2} = \mathscr{T} \quad \text{for all } n \leqq m.$$

Choosing s such that it has no common divisor with r, we find integers $n \leqq n_t$ and $n' \leqq m$ such that $nr + t = n's$. This shows that $\mathscr{S}_t = \mathscr{T}$ for $t = 0, \dots, r-1$. From this our claim follows immediately. □

5.4. Corollary. *Let X_0, $\mathscr{F}$, $\mathscr{L}$ and i be as in Theorem* 5.3, *and let* $\dim(X_0) \leqq 2$.

Then, the set $\mathrm{Ass}_{\mathscr{O}(X_0)}\big(H^i(X, \mathscr{L}^n \otimes_{\mathscr{O}_X} \mathscr{F})\big)^{\leqq 2}$ *is asymptotically stable for $n \to -\infty$.* □

5.5. Theorem. *Let X_0 be essentially of finite type over a field. Let $\mathscr{F}$ be a coherent sheaf of $\mathscr{O}_X$-modules. Let $\mathscr{L}$ be an ample invertible sheaf of $\mathscr{O}_X$-modules and let $i \in \mathbb{N}_0$.*

Then, the set $\mathrm{Ass}_{X_0}\big(\mathscr{R}^i\pi_*(\mathscr{L}^n \otimes_{\mathscr{O}_X} \mathscr{F})\big)^{\leqq 2}$ *is asymptotically stable for $n \to -\infty$.*

Proof. The result is local in the base. So we may assume that X_0 is affine. Now we conclude by Theorem 5.3 as $\mathscr{R}^i\pi_*(\mathscr{L}^n \otimes_{\mathscr{O}_X} \mathscr{F}) \cong H^i(X, \mathscr{L}^n \otimes_{\mathscr{O}_X} \mathscr{F})^{\sim}$.

5.6. Corollary. *Let X_0, $\mathscr{F}$, $\mathscr{L}$ and i be as in Theorem* 5.5, *and let* $\dim(X_0) \leqq 2$.

Then, the set $\mathrm{Ass}_{X_0}\big(\mathscr{R}^i\pi_*(\mathscr{L}^n \otimes_{\mathscr{O}_X} \mathscr{F})\big)$ *is asymptotically stable for $n \to -\infty$.* □

5.7. Example. Let R_0 and R be as in Example 4.3, so that $X_0 := \mathrm{Spec}(R_0)$ is the affine 3-space $\mathbb{A}^3_K$ and $X = \mathrm{Proj}(R) \subseteqq \mathbb{P}^1_{X_0}$ is the projective line over $\mathbb{A}^3_K$. Then the Serre-Grothendieck Correspondence (cf. Proof of Proposition 5.2) yields, that the set

$$\bigcup_{n \leqq 0} \mathrm{Ass}_{\mathscr{O}(X_0)}\big(H^1\big(X, \mathscr{O}_X(n)\big)\big) \text{ is not finite.}$$

Therefore the set

$$\mathrm{Ass}_{\mathscr{O}(X_0)}\big(H^1\big(X, \mathscr{O}_X(n)\big)\big) = \mathrm{Ass}_{X_0}\big(\mathscr{R}^1\pi_*\big(\mathscr{O}_X(1)^n\big)\big)$$

is not asymptotically stable for $n \to -\infty$. This shows, that in a surprisingly simple situation the conclusions of Corollaries 5.4 and 5.6 need not hold over a base scheme X_0 of dimension $\geqq 3$.

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