J. Aust. Math. Soc. **79** (2005), 231–241

OPERATOR ALGEBRAS RELATED TO THOMPSON'S GROUP F

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(Received 24 October 2003; revised 25 November 2004)

Communicated by A. Pryde

Abstract

Let F' be the commutator subgroup of F and let Γ_0 be the cyclic group generated by the first generator of F. We continue the study of the central sequences of the factor $L(F')$, and we prove that the abelian von Neumann algebra $L(\Gamma_0)$ is a strongly singular MASA in $L(F)$. We also prove that the natural action of F on $[0, 1]$ is ergodic and that its ratio set is $\{0\} \cup \{2^k;\ k \in \mathbb{Z}\}$.

2000 *Mathematics subject classification*: primary 37A10; secondary 43A07.

1. Introduction

Thompson's group F is a countable group with infinite conjugacy classes which has remarkable properties, discovered in 1965 and rediscovered later by homotopy theorists. It is known that F contains no non abelian free subgroup, and it is still unknown whether it is amenable or not. However, it shares many properties with amenable groups; for instance, it is inner amenable: see [6, 7].

We recall the two most commonly known descriptions of F. On the one hand, as an abstract group, it has the following presentation:

$$F = \langle x_0, x_1, \dots \mid x_i^{-1} x_n x_i = x_{n+1},\ 0 \le i < n \rangle.$$

On the other hand, it is the group of homeomorphisms of the interval $[0, 1]$ that are piecewise linear, differentiable except at finitely many dyadic rational numbers and such that on intervals of differentiability the derivatives are integral powers of 2. To each of these descriptions corresponds a tool that we present below. We refer to [1, 2] for more details on Thompson's groups F, T and V, and for some results that will be used here. The first one is the existence of a unique *normal form* for every non trivial element of F; it is Corollary-Definition 2.7 of [2].

LEMMA 1.1. *Every non trivial element of F can be expressed in unique normal form $x_{i_1} \cdots x_{i_k} x_{j_m}^{-1} \cdots x_{j_1}^{-1}$, where*

(1) $0 \leq i_1 \leq \cdots \leq i_k, j_1 \leq \cdots \leq j_m$ *and* $i_k \neq j_m$;

(2) *if x_i and x_i^{-1} appear explicitly in the expression for some i, then also does x_{i+1} or x_{i+1}^{-1}.*

The second lemma that will be needed is related to the action of F on $[0, 1]$: see [2, Lemma 4.2].

LEMMA 1.2. *If $0 = a_0 < a_1 < \cdots < a_n = 1$ and $0 = b_0 < b_1 < \cdots < b_n = 1$ are partitions of $[0, 1]$ consisting of dyadic rational numbers, then there exists $g \in F$ such that $g(a_i) = b_i$ for every $i = 0, \ldots, n$. If moreover $a_{i-1} = b_{i-1}$ and $a_i = b_i$ for some $i > 0$, then g can be chosen so that $g(t) = t$ for every $t \in [a_{i-1}, a_i]$.*

In [7], we studied the von Neumann algebras associated with F and with some of its subgroups: we proved not only that the factor $L(F)$ is a McDuff factor (that is, that $L(F)$ is *-isomorphic to the tensor product factor $L(F) \otimes R$ where R denotes the hyperfinite II_1 factor), but that the pair $L(F') \subset L(F)$ has the relative McDuff property: there is a *-isomorphism $\Phi : L(F) \otimes R \to L(F)$ such that its restriction to $L(F') \otimes R$ is still an isomorphism onto the subfactor $L(F')$ associated to the derived subgroup $F' = [F, F]$ of F.

In Section 2, we present further properties of central sequences of $L(F')$: on the one hand, we prove that there exists a sequence of unitary operators $(u_n) \subset L(F')$ such that $(u_n x u_n^*)$ is a central sequence for every $x \in L(F')$, and on the other hand, we show that $L(F')$ is *approximately normal* in $L(F)$ in the sense of [3].

Section 3 is devoted to the study of the von Neumann subalgebra $L(\Gamma_0)$ where Γ_0 is the cyclic subgroup of F generated by the generator x_0: $L(\Gamma_0)$ is a *strongly singular* MASA of $L(F)$ (see [8]) and we show that its Pukanszky invariant is $\{\infty\}$.

Finally, we prove in Section 4 that the natural action of F on $[0, 1]$ is ergodic and we show that its ratio set is $\{0\} \cup \{2^k;\ k \in \mathbb{Z}\}$.

Notation For $g \in F$ viewed as a homeomorphism of $[0, 1]$, we denote by $S(g)$ the set of numbers $t \in [0, 1]$ such that $g(t) \neq t$. We will use the following properties which are very easy to prove:

(S1) $S(g^{-1}) = S(g)$ for every $g \in F$;

(S2) if $t \in S(g)$ then $g(t) \in S(g)$ for every $g \in F$;

(S3) $S(ghg^{-1}) = g(S(h))$ for all $g, h \in F$;

(S4) if $S(g) \cap S(h) = \emptyset$, then $gh = hg$.

Let M be a finite von Neumann algebra and let τ be a normal, normalized, faithful trace on M. We denote by $\|a\|_2 = \tau(a^*a)^{1/2}$ the associated Hilbertian norm.

If Γ is a countable group, let λ denote its left regular representation on $\ell^2(\Gamma)$ defined by:

$$(\lambda(g)\xi)(h) = \xi(g^{-1}h)$$

for all $g, h \in \Gamma$ and $\xi \in \ell^2(\Gamma)$. The bicommutant $\lambda(\Gamma)''$ in the algebra of linear, bounded operators on $\ell^2(\Gamma)$ is a von Neumann algebra denoted by $L(\Gamma)$. It is a finite von Neumann algebra with natural trace τ defined by $\tau(x) = \langle x\delta_e, \delta_e\rangle$, where δ_e is the characteristic function of $\{e\}$. $L(\Gamma)$ is a type II_1 factor if and only if Γ is an icc group. Every operator $x \in L(\Gamma)$ is expressed as a series $\sum_{g\in\Gamma} x(g)\lambda(g)$ where $x(g) = \tau(x\lambda(g^{-1}))$ and $\sum_{g\in\Gamma} |x(g)|^2 = \|x\|_2^2$. If H is a subgroup of Γ, then $L(H)$ is identified in a natural way as a von Neumann subalgebra of $L(\Gamma)$: it is the subset of elements x such that $x(g) = 0$ for every $g \notin H$.

Let M be a type II_1 factor. A *central sequence* in M is a bounded sequence $(x_n) \subset M$ such that $\|x_n y - y x_n\|_2 \to 0$ as $n \to \infty$ for every $y \in M$. We write sometimes $[a, b]$ for $ab - ba$.

2. The subfactor $L(F')$ of $L(F)$

Let F' denote the commutator subgroup of F as in the preceding section. It is known that it is a simple group, and it is the subgroup of F of elements that coincide with the identity in neighbourhoods of 0 and 1. Our first result states that $L(F')$ is *inner asymptotically commutative* in the sense of [10].

PROPOSITION 2.1. *There exists a sequence $(u_n)_{n\geq 1} \subset U(L(F'))$ such that*

$$\lim_{n\to\infty} \|[u_n x u_n^*, y]\|_2 = 0$$

for all $x, y \in L(F')$. In other words, for every $x \in L(F')$, $(u_n x u_n^*)$ is a central sequence of $L(F')$.

We don't know if $L(F)$ itself is inner asymptotically commutative.

The proof rests on the following lemma of Zeller-Meyer:

LEMMA 2.2. *Let G be an icc group. If there exists an increasing sequence (E_n) of subsets of G and a sequence (g_n) of elements of G such that $\bigcup E_n = G$ and $g_n g g_n^{-1} h = h g_n g g_n^{-1}$ for all $g, h \in E_n$ and for every n, then the factor $L(G)$ is inner asymptotically commutative.*

PROOF OF PROPOSITION 2.1. Set, for $n \geq 1$,

$$E_n = \{g \in F';\ S(g) \subset (1/2^n, 1 - 1/2^n)\},$$

so that $E_n \subset E_{n+1}$ and $\bigcup_n E_n = F'$.

Fix $n \geq 1$ and consider the following dyadic partitions of $[0, 1]$:

$$0 = a_0 < a_1 = \frac{1}{2^{n+1}} < a_2 = \frac{1}{2^n} < a_3 = 1 - \frac{1}{2^n} < a_4 = 1 - \frac{1}{2^{n+1}} < a_5 = 1$$

and

$$0 = b_0 < b_1 = \frac{1}{2^{n+1}} < b_2 = 1 - \frac{1}{2^n} < b_3 = 1 - \frac{3}{2^{n+2}} < b_4 = 1 - \frac{1}{2^{n+1}} < b_5 = 1$$

so that b_3 is the mean value of $1 - 1/2^n$ and $1 - 1/2^{n+1}$. By Lemma 1.2, one can find $g_n \in F$ such that $g_n(a_i) = b_i$ for $i = 0, \dots, 5$ and that $g_n(t) = t$ for $t \in [0, a_1] \cup [a_4, 1]$. Thus $g_n \in F'$.

Finally, if $x, y \in E_n$, Property (S3) of Section 1 gives

$$S(g_n x g_n^{-1}) = g_n(S(x)) \subset g_n((a_2, a_3)) \subset (b_2, b_3),$$

and as $S(y) \subset (a_2, a_3)$, Property (S4) yields $g_n x g_n^{-1} y = y g_n x g_n^{-1}$. Lemma 2.2 implies that $L(F')$ is inner asymptotically commutative. □

Before stating the next proposition, let M be a type II_1 factor and let N be a subalgebra of M. N is called *normal* in M if $(N' \cap M)' \cap M = N$. Following [3], we say that N is *approximately normal* in M if

$$N = \{x \in M;\ \|y_n x - x y_n\|_2 \to 0 \text{ for all bounded sequences } (y_n) \subset M \text{ such that } \|y_n y - y y_n\|_2 \to 0 \text{ for all } y \in N\}.$$

PROPOSITION 2.3. *The subfactor $L(F')$ is approximately normal in $L(F)$.*

For future use, denote by $\mathscr{T}$ the set of sequences $(g_n)_{n \geq 1} \subset F'$ such that, for every $\varepsilon > 0$, there exists an integer $l > 0$ such that $S(g_n) \cap [\varepsilon, 1 - \varepsilon] = \emptyset$ for every $n > l$. In fact, if $(g_n) \in \mathscr{T}$, then the sequence $(\lambda(g_n)) \subset L(F')$ is a central sequence (see for instance the proof of [7, Proposition 2.4]).

The proof of Proposition 2.3 rests on the following lemma:

LEMMA 2.4. *Let $h \in F \setminus F'$. Then there exists $(g_n) \in \mathscr{T}$ such that the set $\{g_n h g_n^{-1};\ n \geq 1\}$ is infinite.*

PROOF. Since $h \notin F$, its right derivative at 0 or its left derivative at 1 is equal to 2^k for some non zero integer k. Assume to begin with that there exist $\varepsilon > 0$ and a positive integer k such that $h(t) = 2^k t$ for $t \in [0, \varepsilon]$. Choose an integer $l \geq 1$ big enough so that $3/2^{l+2} \leq \varepsilon$. For $n \geq l$, set $a_n = 1/2^{k+n+1}$, $b_n = 3/2^{k+n+2}$, $c_n = 1/2^{k+n}$ and

$$a_n < b'_n = \frac{3}{2^{k+n+2}} - \frac{1}{2^{k+n+3}} = \frac{5}{2^{k+n+3}} < b_n.$$

There exists $g_n \in F'$ such that $g_n(t) = t$ for $t \in [0, a_n] \cup [c_n, 1]$ and $g_n(b_n) = b'_n$. Thus, setting $g_n = e$ for $n \leq l$, we get that $(g_n) \in \mathscr{T}$. Finally, let us check that if $m > k + n$, then $g_m h g_m^{-1} \neq g_n h g_n^{-1}$. As $S(g_p) \subset (1/2^{k+p+1}, 1/2^{k+p})$, we get, on the one hand,

$$g_n h g_n^{-1}(b'_m) = g_n h(b'_m) = g_n(2^k b'_m) = \frac{5}{2^{m+3}}$$

since $5/2^{m+3} < 1/2^{k+n+1}$. On the other hand, the same kind of argument using $3/2^{m+3} \leq \varepsilon$ shows that

$$g_m h g_m^{-1}(b'_m) = \frac{3}{2^{m+3}} \neq g_n h g_n^{-1}(b'_m).$$

If $h(t) = 2^{-k}t$ near 0 with $k > 0$, then apply the above arguments to h^{-1}. Finally, if $h(t) = t$ near 0, we reduce to the above cases in using the order two automorphism θ of F defined by $\theta(g)(t) = 1 - g(1-t)$: apply the above arguments to $\theta(h)$ and take the sequence $(\theta(g_n)) \in \mathscr{T}$. This ends the proof of the lemma. □

PROOF OF PROPOSITION 2.3. We only need to show that if $x \in L(F)$ is such that $\|[x, \lambda(g_n)]\|_2 \to 0$ as $n \to \infty$ for every sequence $(g_n) \in \mathscr{T}$, then $x \in L(F')$.

Let $h \in F \setminus F'$; we will prove that $x(h) = 0$. By the previous lemma, one can find a sequence $(g_n) \in \mathscr{T}$ such that $g_n h g_n^{-1} \neq g_m h g_m^{-1}$ for all $n \neq m$.

Fix $\varepsilon > 0$. There exists an integer $l \geq 1$ such that $\|x - \lambda(g_n^{-1}) x \lambda(g_n)\|_2 < \varepsilon/2$ for $n \geq l$. Moreover, the series

$$\sum_{n=1}^{\infty} |x(g_n h g_n^{-1})|^2$$

converges. Hence one can find an integer $l' \geq l$ such that $|x(g_n h g_n^{-1})| < \varepsilon/2$ for every $n \geq l'$. Then we have, for any $n \geq l'$,

$$\begin{aligned} |x(h)| &= |\tau(x\lambda(h^{-1}))| \\ &\leq |\tau(x\lambda(h^{-1}) - x\lambda(g_n h^{-1} g_n^{-1}))| + |\tau(x\lambda(g_n h^{-1} g_n^{-1}))| \\ &= |\tau(x\lambda(h^{-1})) - \tau(\lambda(g_n^{-1}) x \lambda(g_n) \lambda(h^{-1}))| + |\tau(x\lambda(g_n h^{-1} g_n^{-1}))| \\ &\leq \|x - \lambda(g_n^{-1}) x \lambda(g_n)\|_2 + |x(g_n h g_n^{-1})| < \varepsilon. \end{aligned}$$

Thus $x(h) = 0$. □

3. A strongly singular MASA in $L(F)$

Let Γ be an icc countable group and let Γ_0 be an abelian subgroup of Γ. When $\Gamma = F$, Γ_0 denotes exclusively the cyclic subgroup generated by $x_0 \in F$.

Recall from [8] that an abelian von Neumann subalgebra A of a type II_1 factor M is *strongly singular* if the following inequality is true for every unitary element $u \in M$:

$$\|E_{uAu^*} - E_A\|_{\infty,2} \geq \|u - E_A(u)\|_2,$$

where E_B denotes the trace-preserving conditional expectation from M onto the von Neumann algebra B, and, for $\phi : M \to M$ linear,

$$\|\phi\|_{\infty,2} = \sup\{\|\phi(x)\|_2;\ \|x\| \leq 1\}.$$

Observe that such an algebra is automatically a MASA of M. When $M = L(\Gamma)$ is a factor associated to an icc group Γ and $A = L(\Gamma_0)$ where Γ_0 is an abelian subgroup of Γ, [8, Lemma 4.1] gives a sufficient condition in order that A be a strongly singular MASA in M; we recall its statement for convenience.

LEMMA 3.1 ([8]). *Let Γ be an icc group with an abelian subgroup Γ_0. Assume that they satisfy the following condition*:

(SS) *If $g_1, \dots, g_m, h_1, \dots, h_n \in \Gamma \setminus \Gamma_0$, then there exists $g_0 \in \Gamma_0$ such that $g_i g_0 h_j \notin \Gamma_0$ for all $i = 1, \dots, m$ and all $j = 1, \dots, n$.*

Then $A = L(\Gamma_0)$ is a strongly singular MASA in $L(\Gamma)$.

It turns out that the abelian subgroup Γ_0 of F satisfies a stronger condition that will be also used to prove an ergodic property for the conditional expectation onto $L(\Gamma_0)$.

LEMMA 3.2. *The pair (Γ_0, F) satisfies the following condition*:

(ST) *If $g_1, \dots, g_m, h_1, \dots, h_n \in F \setminus \Gamma_0$, then there exists a finite subset E of Γ_0 such that $g_i g_0 h_j \notin \Gamma_0$ for all $i = 1, \dots, m$, all $j = 1, \dots, n$ and for every $g_0 \in \Gamma_0 \setminus E$.*

PROOF. Fix $g_1, \dots, g_m, h_1, \dots, h_n \in F \setminus \Gamma_0$. We write $g_i = x_0^{p_i} g_i' x_0^{-q_i}$ and $h_j = x_0^{r_j} h_j' x_0^{-s_j}$ for all i, j, where p_i, q_i, r_j and s_j are non negative integers, and where g_i', h_j' all belong to $F_1 \setminus \{e\}$, where F_1 is the subgroup of F generated by $\{x_1, x_2, \dots\}$. It is the range of the 'shift map' $\phi : F \to F$ defined by $\phi(x_n) = x_{n+1}$ for every $n \geq 0$. See [1, page 369]. Hence $x_0^{-k} g x_0^k = \phi^k(g)$ for every positive integer k and every $g \in F_1$. Using normal forms of g_i' and h_j', observe that if l is a sufficiently large integer, then $\phi^l(g_i')h_j' \notin \Gamma_0$ for all i, j. Thus, if $k > 0$ is large enough and if we put $l_{i,j} = k - q_i + r_j$, we get $g_i x_0^k h_j = x_0^{p_i} g_i' x_0^{l_{i,j}} h_j' x_0^{-s_j} = x_0^{p_i + l_{i,j}} \phi^{l_{i,j}}(g_i') h_j' x_0^{-s_j}$, and we deduce that $g_i x_0^k h_j$ does not belong to Γ_0.

When $k = -k'$ is negative, we have $g_i' x_0^{-k'} h_j' = g_i' \phi^{k'}(h_j') x_0^{-k'}$ for all i, j, which gives the same conclusion. □

Similarly, it is easy to see that if Γ is the free group F_N of rank $N \geq 2$ on free generators $a_1, \dots, a_N$ and if Γ_0 is the subgroup generated by some a_i, then the pair (Γ_0, Γ) satisfies condition (ST).

PROPOSITION 3.3. *Assume that Γ_0 is an abelian subgroup of an icc group Γ and that the pair (Γ_0, Γ) satisfies condition* (ST). *Set $A = L(\Gamma_0)$ and $M = L(\Gamma)$. Then A is a strongly singular MASA in M and the conditional expectation E_A satisfies*:

$$E_A(x) = w - \lim_{g_0 \to \infty} \lambda(g_0) x \lambda(g_0^{-1})$$

for every $x \in M$.

PROOF. The first assertion follows from Lemma 3.1. In order to prove the second one, fix x and $y \in M$ such that $\{g \in \Gamma;\ x(g) \neq 0\}$ and $\{g \in \Gamma;\ y(g) \neq 0\}$ are finite. We will prove that there exists a finite subset E of Γ_0 such that

$$\tau(E_A(x) y^*) = \tau(\lambda(g_0) x \lambda(g_0^{-1}) y^*)$$

for every $g_0 \in \Gamma_0 \setminus E$. We write $C = \{g \in \Gamma \setminus \Gamma_0; x(g) \neq 0\}$ and $D = \{g \in \Gamma \setminus \Gamma_0; y(g) \neq 0\}$, which are finite sets, and we decompose x and y as $x = E_A(x) + x'$ and $y = E_A(y) + y'$ so that $x' = \sum_{g \in C} x(g)\lambda(g)$ and $y' = \sum_{g \in D} y(g)\lambda(g)$ are orthogonal to A.

It is easy to see that condition (ST) implies the existence of a finite subset E of Γ_0 such that $g_0 C g_0^{-1} \cap D = \emptyset$ for every $g_0 \in \Gamma_0 \setminus E$. Take $g_0 \in \Gamma_0 \setminus E$; since Γ_0 is abelian, we have $\lambda(g_0) E_A(x) \lambda(g_0^{-1}) = E_A(x)$ and similarly for y. Thus

$$\begin{aligned} \tau(\lambda(g_0) x \lambda(g_0^{-1}) y^*) &= \tau(E_A(x) y^*) + \tau(\lambda(g_0) x' \lambda(g_0^{-1}) y^*) \\ &= \tau(E_A(x) y^*) + \tau(x' E_A(y^*)) + \tau(x' \lambda(g_0^{-1}) y'^* \lambda(g_0)) \\ &= \tau(E_A(x) y^*) \end{aligned}$$

since $\tau(x' E_A(y^*)) = 0$ by orthogonality, and

$$\begin{aligned} \tau(x' \lambda(g_0^{-1}) y'^* \lambda(g_0)) &= \tau\left(\sum_{h,k \in \Gamma \setminus \Gamma_0} x(h) \overline{y(k)} \lambda(h g_0^{-1} k^{-1} g_0) \right) \\ &= \sum_{h \in C} x(h) \overline{y(g_0 h g_0^{-1})} = 0 \end{aligned}$$

because $g_0 C g_0^{-1} \cap D = \emptyset$. □

Our last result about $L(\Gamma_0) \subset L(F)$ is the computation of its Pukanszky invariant which was suggested to us by G. Robertson. Let A be a MASA of some type II_1 factor M. Denote by $\mathscr{A}$ the type I von Neumann algebra generated by $A \cup A'$ in the algebra $B(L^2(M))$, where A' is the commutant of A. The orthogonal projection e_A of $L^2(M)$ onto $L^2(A)$ lies in the centre of $\mathscr{A}'$ and the reduced algebra $\mathscr{A}'e_A$ is abelian. The reduced von Neumann algebra $\mathscr{A}'(1-e_A)$ decomposes as a direct sum $\mathscr{A}'(1-e_A) = \mathscr{A}'_{n_1} \oplus \mathscr{A}'_{n_2} \oplus \dots$ of homogeneous type I_{n_i} algebras, where $1 \le n_1 < n_2 < \cdots \le \infty$. Then the *Pukanszky invariant* of the pair $A \subset M$ is the set $\{n_1, n_2, \dots\}$.

Let us recall [9, Proposition 3.6] which gives a way to compute Pukanszky invariant for pairs $L(\Gamma_0) \subset L(\Gamma)$ as above.

PROPOSITION 3.4 ([9]). *Suppose that Γ_0 is an abelian subgroup of an icc group Γ such that $L(\Gamma_0)$ is a MASA in $L(\Gamma)$. If $g^{-1}\Gamma_0 g \cap \Gamma_0 = \{e\}$ for every $g \in \Gamma \setminus \Gamma_0$, then the Pukanszky invariant of the pair $L(\Gamma_0) \subset L(\Gamma)$ is the set reduced to the number n of double classes $\Gamma_0 g \Gamma_0$.*

PROPOSITION 3.5. *The Pukanszky invariant of the pair $L(\Gamma_0) \subset L(F)$ is $\{\infty\}$.*

PROOF. There exists a homomorphism $\psi : F \to \mathbb{Z}$ such that $\psi(x_n) = 1$ for every $n \ge 0$ because the defining relations of F are homogeneous. If there would exist $g \in F \setminus \Gamma_0$ and $k, l \in \mathbb{Z}$ such that $g x_0^k g^{-1} = x_0^l$, then, applying ψ on both sides gives $k = l$ which can be assumed positive. Let us write $g = x_0^p g' x_0^{-q}$ with $g' \in F_1 \setminus \{e\}$ as in the proof of Lemma 2.4. We would get $g' x_0^k = x_0^k g'$, thus $\phi^k(g') = g'$, but this would imply that $g' = e$, hence that $g \in \Gamma_0$, but this is a contradiction.

Finally, set, for every integer $n > 0 : g_n = x_1 \cdots x_n$. Then it is easy to see that the double classes $\Gamma_0 g_n \Gamma_0$ are pairwise distinct. □

4. The natural action of F and the associated Krieger factor

Let Γ be a group acting (on the left) in a measure class preserving way on a standard probability space (Ω, μ). Recall that the *full group* $[\Gamma]$ of the action of Γ on Ω is the group of all automorphisms T of Ω such that $T\omega \in \Gamma\omega$ for μ-a.e. $\omega \in \Omega$. We also set

$$[\Gamma]_0 = \{T \in [\Gamma];\ \mu \circ T = \mu\},$$

which is the subgroup of elements of $[\Gamma]$ that preserve μ. The *ratio set* of the action of Γ on Ω is the set $r(\Gamma)$ of all numbers $\lambda \ge 0$ such that, for every $\varepsilon > 0$ and for every Borel subset A with positive measure, there exists a subset B of A with positive measure and $g \in \Gamma$ such that $gB \subset A$ and

$$\left| \frac{d\mu \circ g}{d\mu}(\omega) - \lambda \right| < \varepsilon \quad \forall \omega \in B.$$

Then $r(\Gamma)$ is a closed subset of $[0, \infty)$ and $r(\Gamma) \setminus \{0\}$ is a closed subgroup of the multiplicative group $\mathbb{R}_+^*$. Moreover, one has: $r(\Gamma) = r([\Gamma])$.

We consider here the natural action of F on the interval $[0, 1]$ gifted with Lebesgue measure μ. Our goal is to prove that the action of F is ergodic and to compute its ratio set.

PROPOSITION 4.1. *The action of F on $[0, 1]$ is ergodic and its ratio set is*

$$r(F) = \{0\} \cup \{2^k;\ k \in \mathbb{Z}\}.$$

Let R_F be the equivalence relation on $[0, 1]$ defined by the action of F: if $s, t \in [0, 1]$, then the pair (s, t) belongs to R_F if and only if s and t are in the same F-orbit. Let $M(R_F)$ be the associated Feldman-Moore factor [5]; it generalizes the group measure space construction of Murray and von Neumann for not necessarily free actions. Then the next result follows immediately from Proposition 4.1 and from [5, Proposition 2.11].

COROLLARY 4.2. *$M(R_F)$ is a factor of type* $\mathrm{III}_{1/2}$.

The proof of Proposition 4.1 is inspired by [4, Section 2].

Let K be the following group of bijections of $[0, 1]$: a bijection φ from $[0, 1]$ to itself belongs to K if and only if there exists a partition $0 < a_1 < \cdots < a_n < 1$ of $[0, 1]$ into dyadic rational numbers such that

(K1) $\varphi(t) = t$ for every $t \in [0, a_1) \cup [a_n, 1]$;
(K2) for every $1 \le j \le n - 1$, there exists a dyadic rational number α_j such that $\varphi(t) = t + \alpha_j$ for every $t \in [a_j, a_{j+1})$.

LEMMA 4.3. *K is a subgroup of $[F]_0$. In particular, $\mu \circ \varphi = \mu$ for every $\varphi \in K$.*

PROOF. It suffices to prove that, if a, b and α are rational dyadic numbers such that $0 < a < b < 1$ and $0 < a + \alpha < b + \alpha < 1$, then there exists $f \in F$ such that $f(t) = t + \alpha$ for every $a \le t < b$. We apply Lemma 1.2 with $a_1 = a$, $a_2 = b$ and $b_1 = a + \alpha$, $b_2 = b + \alpha$ (and $a_0 = b_0 = 0$, $a_3 = b_3 = 1$): there exists $g \in F$ such that $g(a_j) = b_j$ for $j = 0, \ldots, 3$. If $g(t) = t + \alpha$ for $t \in [a, b)$, then set $f = g$. If not, set $f(t) = g(t)$ for $t \in [0, a) \cup [b, 1]$ and $f(t) = t + \alpha$ for $a \le t < b$. Then $f \in F$. □

PROOF OF PROPOSITION 4.1. In order to prove that the action of F is ergodic, it suffices to show that the action of K is. Indeed, Lemma 4.3 implies thus that the action of the full group $[F]$ is ergodic, and [4, Lemma 2.8] applies to show that the action of F is, too. We argue as in the proof of [4, Lemma 2.1]. Thus, let $X_0 \subset [0, 1]$

be a Borel set such that $\varphi X_0 = X_0$ for every $\varphi \in K$ and that $\mu(X_0) > 0$. Define a measure ν on $[0, 1]$ by

$$\nu(B) = \frac{\mu(B \cap X_0)}{\mu(X_0)}$$

for every Borel set B. We have for every $\varphi \in K$ and every Borel set B

$$\begin{aligned}\nu(\varphi B) &= \frac{\mu((\varphi B) \cap X_0)}{\mu(X_0)} = \frac{\mu(B \cap \varphi^{-1} X_0)}{\mu(X_0)} \\ &\leq \frac{\mu(B \cap X_0)}{\mu(X_0)} + \frac{\mu(B \cap (\varphi^{-1} X_0 \setminus X_0))}{\mu(X_0)} \\ &= \frac{\mu(B \cap X_0)}{\mu(X_0)} = \nu(B).\end{aligned}$$

Hence $\nu(\varphi B) = \nu(B)$ for every φ. In particular, one has $\nu([a+\alpha, b+\alpha)) = \nu([a, b))$ for all dyadic rational numbers a, b and α such that $[a, b] \cup [a + \alpha, b + \alpha] \subset [0, 1]$, and this gives

$$\nu\left(\left[\frac{l}{2^n}, \frac{l+1}{2^n}\right)\right) = \frac{1}{2^n}$$

for every positive integer n and every integer $0 \leq l \leq n$. Uniqueness of μ implies that $\nu = \mu$, and that $\mu(X_0) = 1$. This proves ergodicity of the action of K.

It remains to compute $r(F)$. Denote by Γ the group generated by F and K. One has $[\Gamma] = [F]$, since $K \subset [F]$, and $r(\Gamma) = r(F)$.

As $(d\mu \circ g)(t)/d\mu \in \{2^k;\ k \in \mathbb{Z}\}$ for μ-a.e. $t \in [0, 1]$ and for every $g \in F$, one has $r(F) \subset \{0\} \cup \{2^k;\ k \in \mathbb{Z}\}$. As $r(F)$ is closed, the proof will be complete if we show that $2^k \in r(F)$ for every integer k. Then fix such a k and choose $g \in F$ and dyadic rational numbers $0 < a < b < 1$ such that

$$\frac{d\mu \circ g}{d\mu}(t) = \frac{dg}{dt}(t) = 2^k$$

for every $t \in (a, b)$. Let $A \subset [0, 1]$ be a Borel set with positive measure. As the action of K is ergodic, there exist φ and $\psi \in K$ such that the Borel set

$$B = \{t \in A;\ \psi(t) \in (a, b) \text{ and } \varphi g \psi(t) \in A\}$$

has positive measure (see the proof of [4, Proposition 3.3]). Set $h = \varphi g \psi \in \Gamma$. One has $B \cup hB \subset A$ and

$$\frac{d\mu \circ h}{d\mu}(t) = \frac{d\mu \circ g}{d\mu}(\psi(t)) = 2^k$$

for every $t \in B$ since $\psi(t) \in (a, b)$. Hence $2^k \in r(F)$. □

REMARK. In fact, condition (K1) shows that K is a subgroup of the full group of the commutator subgroup F'. Hence the latter acts ergodically on [0, 1] as well, and it has the same ratio set as F. However, it gives the same equivalence relation on [0, 1], and thus the same factor $M(R_F)$.

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