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Global Action–Angle Variables for the Periodic Toda Lattice

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In this paper, we construct *global* action–angle variables for the periodic Toda lattice.

1 Introduction

Consider the Toda lattice with period N ($N \geq 2$),

$$\dot{q}_n = \partial_{p_n} H, \quad \dot{p}_n = -\partial_{q_n} H$$

for $n \in \mathbb{Z}$, where the (real) coordinates $(q_n, p_n)_{n\in\mathbb{Z}}$ satisfy $(q_{n+N}, p_{n+N}) = (q_n, p_n)$ for any $n \in \mathbb{Z}$ and the Hamiltonian H_{Toda} is given by

$$H_{Toda} = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + \alpha^2 \sum_{n=1}^{N} e^{q_n - q_{n+1}}, \tag{1}$$

where α is a positive parameter, $\alpha > 0$. For the standard Toda lattice, $\alpha = 1$. The Toda lattice was introduced by Toda [18] and studied extensively in the sequel. It is an important model for an integrable system of N particles in one space dimension with nearest neighbor interaction and belongs to the family of lattices introduced and numerically investigated by Fermi, Pasta, and Ulam in their seminal paper [3]. To prove the integrability

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of the Toda lattice, Flaschka introduced in [4] the (noncanonical) coordinates

$$b_n := -p_n \in \mathbb{R}, \quad a_n := \alpha e^{\frac{1}{2}(q_n - q_{n+1})} \in \mathbb{R}_{>0} \quad (n \in \mathbb{Z}).$$

These coordinates describe the motion of the Toda lattice relative to the center of mass. Note that the total momentum is conserved by the Toda flow, hence any trajectory of the center of mass is a straight line.

In these coordinates, the Hamiltonian H_{Toda} takes the simple form

$$H = \frac{1}{2}\sum_{n=1}^{N} b_n^2 + \sum_{n=1}^{N} a_n^2,$$

and the equations of motion are

$$\begin{cases} \dot{b}_n = a_n^2 - a_{n-1}^2 \\ \dot{a}_n = \frac{1}{2} a_n (b_{n+1} - b_n) \end{cases} \qquad (n \in \mathbb{Z}). \tag{2}$$

Note that $(b_{n+N}, a_{n+N}) = (b_n, a_n)$ for any $n \in \mathbb{Z}$, and $\prod_{n=1}^{N} a_n = \alpha^N$. Hence we can identify the sequences $(b_n)_{n\in\mathbb{Z}}$ and $(a_n)_{n\in\mathbb{Z}}$ with the vectors $(b_n)_{1\le n\le N} \in \mathbb{R}^N$ and $(a_n)_{1\le n\le N} \in \mathbb{R}^N_{>0}$. Our aim is to study the normal form of the system of equations (2) on the phase space

$$\mathcal{M} := \mathbb{R}^N \times \mathbb{R}^N_{>0}.$$

This system is Hamiltonian with respect to the nonstandard Poisson structure $J \equiv J_{b,a}$, defined at a point $(b, a) = ((b_n, a_n)_{1\le n\le N}$ by

$$J = \begin{pmatrix} 0 & A \\ -{}^t A & 0 \end{pmatrix}, \tag{3}$$

where A is the b-independent $N \times N$-matrix

$$A = \frac{1}{2}\begin{pmatrix} a_1 & 0 & \dots & 0 & -a_N \\ -a_1 & a_2 & 0 & \ddots & 0 \\ 0 & -a_2 & a_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -a_{N-1} & a_N \end{pmatrix}. \tag{4}$$

The Poisson bracket corresponding to equation (3) is then given by

$$\begin{aligned}\{F, G\}_J(b, a) &= \langle(\nabla_b F, \nabla_a F),\ J(\nabla_b G, \nabla_a G)\rangle_{\mathbb{R}^{2N}} \\ &= \langle \nabla_b F, A\nabla_a G\rangle_{\mathbb{R}^N} - \langle \nabla_a F, A^t\, \nabla_b G\rangle_{\mathbb{R}^N},\end{aligned} \tag{5}$$

where $F, G \in C^1(\mathcal{M})$ and where ∇_b and ∇_a denote the gradients with respect to the N-vectors $b = (b_1, \dots, b_N)$ and $a = (a_1, \dots, a_N)$, respectively. Therefore, equations (2) can alternatively be written as $\dot{b}_n = \{b_n, H\}_J$, $\dot{a}_n = \{a_n, H\}_J$ $(1 \le n \le N)$. Further note that

$$\{b_n, a_n\}_J = \frac{a_n}{2}; \quad \{b_{n+1}, a_n\}_J = -\frac{a_n}{2}, \tag{6}$$

while $\{b_n, a_k\}_J = 0$ for any n, k with $n \notin \{k, k+1\}$.

Since the matrix A defined by equation (4) has rank $N-1$, the Poisson structure J is degenerate. It admits the two Casimir functions

$$C_1 := -\frac{1}{N}\sum_{n=1}^N b_n \quad \text{and} \quad C_2 := \left(\prod_{n=1}^N a_n\right)^{\frac{1}{N}} \tag{7}$$

whose gradients $\nabla_{b,a} C_i = (\nabla_b C_i, \nabla_a C_i)$ $(i = 1, 2)$, given by

$$\nabla_b C_1 = -\frac{1}{N}(1, \dots, 1), \qquad \nabla_a C_1 = 0, \tag{8}$$

$$\nabla_b C_2 = 0, \qquad \nabla_a C_2 = \frac{C_2}{N}\left(\frac{1}{a_1}, \dots, \frac{1}{a_N}\right), \tag{9}$$

are linearly independent at each point (b, a) of $\mathcal{M}$. (A smooth function $C : \mathcal{M} \to \mathbb{R}$ is a Casimir function for J if $\{C, \cdot\}_J \equiv 0$.)

The main result of this paper is the following one.

Theorem 1.1. The periodic Toda lattice admits globally defined action–angle variables. More precisely,

(i) There exist real analytic functions $(I_n)_{1 \le n \le N-1}$ on $\mathcal{M}$ which are pairwise in involution and which Poisson commute with the Toda Hamiltonian H and the two Casimir functions C_1, C_2, i.e. for any $1 \le m, n \le N-1$, $i = 1, 2$,

$$\{I_m, I_n\}_J = 0 \quad \text{on } \mathcal{M}$$

and

$$\{H, I_n\}_J = 0 \quad \text{and} \quad \{C_i, I_n\}_J = 0 \quad \text{on } \mathcal{M}.$$

(ii) For any $1 \leq n \leq N-1$, there exist a real analytic submanifold D_n of codimension 2 and a function $\theta_n : \mathcal{M} \setminus D_n \to \mathbb{R}$, defined mod 2π and real analytic when considered mod π, so that on $\mathcal{M} \setminus \bigcup_{n=1}^{N-1} D_n$, $(\theta_n)_{1 \leq n \leq N-1}$ and $(I_n)_{1 \leq n \leq N-1}$ are conjugate variables. More precisely, for any $1 \leq m, n \leq N-1$, $i = 1, 2$

$$\{I_m, \theta_n\}_J = \delta_{mn} \quad \text{and} \quad \{C_i, \theta_n\}_J = 0 \quad \text{on } \mathcal{M} \setminus D_n$$

and

$$\{\theta_m, \theta_n\}_J = 0 \quad \text{on } \mathcal{M} \setminus (D_m \cup D_n).$$

□

Let $\mathcal{M}_{\beta,\alpha} := \{(b, a) \in \mathbb{R}^{2N} : (C_1, C_2) = (\beta, \alpha)\}$ denote the level set of (C_1, C_2) for $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$. Note that $(-\beta 1_N, \alpha 1_N) \in \mathcal{M}_{\beta,\alpha}$, where $1_N = (1, \ldots, 1) \in \mathbb{R}^N$. As the gradients $\nabla_{b,a} C_1$ and $\nabla_{b,a} C_2$ are linearly independent everywhere on $\mathcal{M}$, the sets $\mathcal{M}_{\beta,\alpha}$ are (real analytic) submanifolds of $\mathcal{M}$ of codimension 2. Furthermore, the Poisson structure J, restricted to $\mathcal{M}_{\beta,\alpha}$, becomes nondegenerate everywhere on $\mathcal{M}_{\beta,\alpha}$ and therefore induces a symplectic structure $\nu_{\beta,\alpha}$ on $\mathcal{M}_{\beta,\alpha}$. In this way, we obtain a symplectic foliation of $\mathcal{M}$ with $\mathcal{M}_{\beta,\alpha}$ being the symplectic leaves.

Corollary 1.2. On each symplectic leaf $\mathcal{M}_{\beta,\alpha}$, the action variables $(I_n)_{1 \leq n \leq N-1}$ are a maximal set of functionally independent integrals in involution of the periodic Toda lattice. □

In subsequent work [9], we will use Theorem 1.1 to construct *global* Birkhoff coordinates for the periodic Toda lattice. More precisely, we introduce the model space $\mathcal{P} := \mathbb{R}^{2(N-1)} \times \mathbb{R} \times \mathbb{R}_{>0}$ endowed with the degenerate Poisson structure J_0 whose symplectic leaves are $\mathbb{R}^{2(N-1)} \times \{\beta\} \times \{\alpha\}$, endowed with the standard Poisson structure, and prove the following theorem.

Theorem 1.3. There exists a real analytic, canonical diffeomorphism

$$\begin{aligned} \Omega : (\mathcal{M}, J) &\to (\mathcal{P}, J_0) \\ (b, a) &\mapsto ((x_n, y_n)_{1 \leq n \leq N-1}, C_1, C_2) \end{aligned}$$

such that the coordinates $(x_n, y_n)_{1 \leq n \leq N-1}, C_1, C_2$ are global Birkhoff coordinates for the periodic Toda lattice, i.e. $(x_n, y_n)_{1 \leq n \leq N-1}$ are canonical coordinates, C_1, C_2 are the Casimirs, and the transformed Toda Hamiltonian $\hat{H} = H \circ \Omega^{-1}$ is a function of the actions $(I_n)_{1 \leq n \leq N-1}$ and C_1, C_2 alone. □

In [10], we used Theorem 1.3 to obtain a KAM theorem for Hamiltonian perturbations of the periodic Toda lattice.

Related work. Theorem 1.1 and Theorem 1.3 improve on earlier work on the normal form of the periodic Toda lattice in [1, 2]. In particular, we construct global Birkhoff coordinates on all of $\mathcal{M}$ instead of a single symplectic leaf and show that techniques recently developed for treating the KdV equation (cf. [11, 12]) and the defocusing NLS equation (cf. [8, 15]) can also be applied for the Toda lattice.

Outline of the paper. In Section 2, we review the Lax pair of the periodic Toda lattice and collect some auxiliary results on the spectrum of the Jacobi matrix $L(b,a)$ associated to an element $(b,a) \in \mathcal{M}$. In Section 3, we study the action variables $(I_n)_{1\le n\le N-1}$, and in Section 4 we define the angle variables $(\theta_n)_{1\le n\le N-1}$ on $\mathcal{M} \setminus \cup_{n=1}^{n} D_n$ using holomorphic differentials defined on the hyperelliptic Riemann surface associated to the spectrum of $L(b,a)$. In Sections 5 and 6, we establish formulas of the gradients of the actions and angles in terms of products of fundamental solutions and prove orthogonality relations between such products which are then used in Section 7 to show that $(I_n)_{1\le n\le N-1}$ and $(\theta_n)_{1\le n\le N-1}$ are canonical variables and to prove Theorem 1.1 and Corollary 1.2.

2 Preliminaries

It is well known (cf. [18]) that the system (2) admits a Lax pair formulation $\dot{L} = \frac{\partial L}{\partial t} = [B, L]$, where $L \equiv L^+(b,a)$ is the periodic Jacobi matrix defined by

$$L^{\pm}(b,a) := \begin{pmatrix} b_1 & a_1 & 0 & \dots & \pm a_N \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ \pm a_N & \dots & 0 & a_{N-1} & b_N \end{pmatrix}, \tag{10}$$

and B the skew-symmetric matrix

$$B = \begin{pmatrix} 0 & a_1 & 0 & \dots & -a_N \\ -a_1 & 0 & a_2 & \ddots & \vdots \\ 0 & -a_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ a_N & \dots & 0 & -a_{N-1} & 0 \end{pmatrix}.$$

Hence the flow of $\dot{L} = [B, L]$ is isospectral.

Proposition 2.1. For a solution $\big(b(t), a(t)\big)$ of the periodic Toda lattice (2), the eigenvalues $(\lambda_j^+)_{1\le j\le N}$ of $L\big(b(t), a(t)\big)$ are conserved quantities. □

Let us now collect a few results from [16] and [18] of the spectral theory of Jacobi matrices needed in the sequel. Denote by $\mathcal{M}^{\mathbb{C}}$ the complexification of the phase space $\mathcal{M}$,

$$\mathcal{M}^{\mathbb{C}} = \{(b, a) \in \mathbb{C}^{2N} : \operatorname{Re} a_j > 0 \quad \forall\, 1 \le j \le N\}.$$

For $(b, a) \in \mathcal{M}^{\mathbb{C}}$, we consider for any complex number λ the difference equation

$$(R_{b,a}\, y)(k) = \lambda y(k) \quad (k \in \mathbb{Z}), \tag{11}$$

where $y(\cdot) = y(k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ and $R_{b,a}$ is the difference operator

$$R_{b,a} = a_{k-1} S^{-1} + b_k S^0 + a_k S^1 \tag{12}$$

with S^m denoting the shift operator of order $m \in \mathbb{Z}$, that is,

$$(S^m y)(k) = y(k + m) \quad \text{for} \quad k \in \mathbb{Z}.$$

Fundamental solutions. The two fundamental solutions $y_1(\cdot, \lambda)$ and $y_2(\cdot, \lambda)$ of equation (11) are defined by the standard initial conditions $y_1(0, \lambda) = 1$, $y_1(1, \lambda) = 0$, and $y_2(0, \lambda) = 0$, $y_2(1, \lambda) = 1$. They satisfy the *Wronskian identity*

$$W(n) := y_1(n, \lambda) y_2(n + 1, \lambda) - y_1(n + 1, \lambda) y_2(n, \lambda) = \frac{a_N}{a_n}. \tag{13}$$

Note that for $n = N$ one gets

$$W(N) = 1. \tag{14}$$

For each $k \in \mathbb{N}$, $y_i(k, \lambda, b, a)$, $i = 1, 2$, is a polynomial in λ of degree at most $k - 1$ and depends real analytically on (b, a) (see [16]). In particular, one easily verifies that $y_2(N + 1, \lambda, b, a)$ is a polynomial in λ of degree N with leading coefficient α^{-N}.

Wronskian. More generally, one defines for any two sequences $(v(n))_{n\in\mathbb{Z}}$ and $(w(n))_{n\in\mathbb{Z}}$ the Wronskian sequence $\big(W(n)\big)_{n\in\mathbb{Z}} = \big(W(v, w)(n)\big)_{n\in\mathbb{Z}}$ by

$$W(n) := v(n) w(n + 1) - v(n + 1) w(n).$$

Let us recall the following properties of the Wronskian, which can be easily verified.

Lemma 2.2.

(i) If y and z are solutions of equation (11) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively, then $W = W(y, z)$ satisfies for any $k \in \mathbb{Z}$

$$a_k W(k) = a_{k-1} W(k-1) + (\lambda_2 - \lambda_1) y(k) z(k). \tag{15}$$

(ii) If $y(\cdot, \lambda)$ is a one-parameter family of solutions of equation (11) which is continuously differentiable with respect to the parameter λ and $\dot{y}(k, \lambda) := \frac{\partial}{\partial \lambda} y(k, \lambda)$, then $W = W(y, \dot{y})$ satisfies for any $k \in \mathbb{Z}$

$$a_k W(k) = a_{k-1} W(k-1) + y(k, \lambda)^2. \tag{16}$$

□

Discriminant. We denote by $\Delta(\lambda) \equiv \Delta(\lambda, b, a)$ the *discriminant* of equation (11), defined by

$$\Delta(\lambda) := y_1(N, \lambda) + y_2(N+1, \lambda). \tag{17}$$

In the sequel, we will often write Δ_λ for $\Delta(\lambda)$. Note that $y_2(N+1, \lambda)$ is a polynomial in λ of degree N with leading term $\alpha^{-N}\lambda^N$, whereas $y_1(N, \lambda)$ is a polynomial in λ of degree less than N, hence $\Delta(\lambda, b, a)$ is a polynomial in λ of degree N with leading term $\alpha^{-N}\lambda^N$, and it depends real analytically on (b, a) (see e.g. [18]). According to Floquet's Theorem (see e.g. [17]), for $\lambda \in \mathbb{C}$ given, equation (11) admits a periodic or antiperiodic solution of period N if the discriminant $\Delta(\lambda)$ satisfies $\Delta(\lambda) = +2$ or $\Delta(\lambda) = -2$, respectively. (These solutions correspond to eigenvectors of L^+ or L^-, respectively, with $L^\pm$ defined by equation (10).) It turns out to be more convenient to combine these two cases by considering the periodic Jacobi matrix $Q \equiv Q(b, a)$ of size $2N$ defined by

$$Q = \left(\begin{array}{cccc|cccc}
b_1 & a_1 & \dots & 0 & 0 & \dots & 0 & a_N \\
a_1 & b_2 & \ddots & \vdots & 0 & \dots & & 0 \\
\vdots & \ddots & \ddots & a_{N-1} & \vdots & & & \vdots \\
0 & \ddots & a_{N-1} & b_N & a_N & \dots & 0 & 0 \\
\hline
0 & \dots & 0 & a_N & b_1 & a_1 & \dots & 0 \\
0 & \dots & & 0 & a_1 & b_2 & \ddots & \vdots \\
\vdots & & & \vdots & \vdots & \ddots & \ddots & a_{N-1} \\
a_N & \dots & 0 & 0 & 0 & \ddots & a_{N-1} & b_N
\end{array}\right).$$

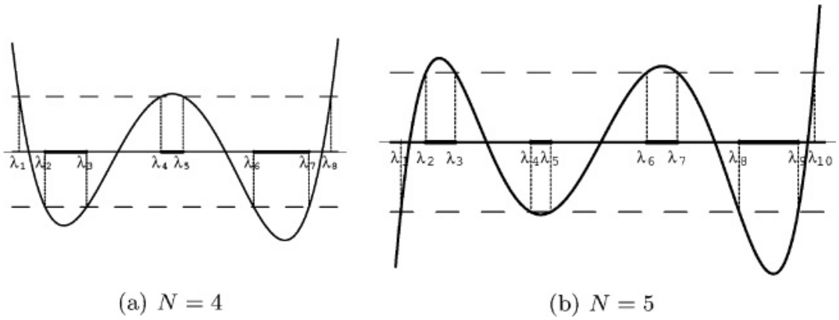

Fig. 1. Examples of the discriminant $\Delta(\lambda)$.

Then the spectrum of the matrix Q is the union of the spectra of the matrices L^+ and L^- and therefore the zero set of the polynomial $\Delta_\lambda^2 - 4$. The function $\Delta_\lambda^2 - 4$ is a polynomial in λ of degree $2N$ and admits a product representation

$$\Delta_\lambda^2 - 4 = \alpha^{-2N} \prod_{j=1}^{2N} (\lambda - \lambda_j). \tag{18}$$

The factor α^{-2N} in equation (18) comes from the above-mentioned fact that the leading term of $\Delta(\lambda)$ is $\alpha^{-N}\lambda^N$.

For any $(b, a) \in \mathcal{M}$, the matrix Q is symmetric and hence the eigenvalues $(\lambda_j)_{1 \le j \le 2N}$ of Q are real. When listed in increasing order and with their algebraic multiplicities, they satisfy the following relations (cf. [16]):

$$\lambda_1 < \lambda_2 \le \lambda_3 < \lambda_4 \le \lambda_5 < \cdots \lambda_{2N-2} \le \lambda_{2N-1} < \lambda_{2N}.$$

As explained above, the λ_j are periodic or antiperiodic eigenvalues of L and thus eigenvalues of L^+ or L^- according to whether $\Delta(\lambda) = 2$ or $\Delta(\lambda) = -2$. One has (cf. [16])

$$\Delta(\lambda_1) = (-1)^N \cdot 2, \quad \Delta(\lambda_{2n}) = \Delta(\lambda_{2n+1}) = (-1)^{n+N} \cdot 2, \quad \Delta(\lambda_{2N}) = 2. \tag{19}$$

See Fig. 1 for an illustration in the cases $N = 4$ and $N = 5$.

Since Δ_λ is a polynomial of degree N, $\dot{\Delta}_\lambda \equiv \dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$ is a polynomial of degree $N - 1$, and it admits a product representation of the form

$$\dot{\Delta}_\lambda = N\alpha^{-N} \prod_{k=1}^{N-1} (\lambda - \dot{\lambda}_k). \tag{20}$$

The zeroes $(\dot{\lambda}_n)_{1\le n\le N-1}$ of $\dot{\Delta}_\lambda$ satisfy $\lambda_{2n} \le \dot{\lambda}_n \le \lambda_{2n+1}$ for any $1 \le n \le N-1$. The intervals $(\lambda_{2n}, \lambda_{2n+1})$ are referred to as the *nth spectral gap* and $\gamma_n := \lambda_{2n+1} - \lambda_{2n}$ as the *nth gap length*. Note that $|\Delta(\lambda)| > 2$ on the spectral gaps. We say that the nth gap is *open* if $\gamma_n > 0$ and *collapsed* otherwise. The set of elements $(b, a) \in \mathcal{M}$ for which the nth gap is collapsed is denoted by D_n,

$$D_n := \{(b, a) \in \mathcal{M} : \gamma_n = 0\}. \tag{21}$$

By writing the condition $\gamma_n = 0$ as $\gamma_n^2 = 0$ and exploiting the fact that γ_n^2 (unlike γ_n) is a real analytic function on $\mathcal{M}$, it can be shown as in [12] that D_n is a real analytic submanifold of $\mathcal{M}$ of codimension 2.

Isolating neighborhoods. Let $(b, a) \in \mathcal{M}$ be given. The strict inequality $\lambda_{2n-1} < \lambda_{2n}$ guarantees the existence of a family of mutually disjoint open subsets $(U_n)_{1\le n\le N-1}$ of $\mathbb{C}$, so that for any $1 \le n \le N-1$, U_n is a neighborhood of the closed interval $[\lambda_{2n}, \lambda_{2n+1}]$. Such a family of neighborhoods is referred to as a family of *isolating neighborhoods* (for (b, a)).

In the case where $(b, a) \in \mathcal{M}^{\mathbb{C}}$, we list the eigenvalues $(\lambda_j)_{1\le j\le 2N}$ in lexicographic ordering $\lambda_1 \prec \lambda_2 \prec \lambda_3 \prec \cdots \prec \lambda_{2N}$.

The lexicographic ordering $a \prec b$ for complex numbers a and b is defined by

$$a \prec b \quad :\Longleftrightarrow \quad \begin{cases} \operatorname{Re} a < \operatorname{Re} b \\ \text{or} \\ \operatorname{Re} a = \operatorname{Re} b \text{ and } \operatorname{Im} a \le \operatorname{Im} b. \end{cases} \tag{22}$$

We then extend the gap lengths γ_n to all of $\mathcal{M}^{\mathbb{C}}$ by

$$\gamma_n := \lambda_{2n+1} - \lambda_{2n} \quad (1 \le n \le N-1)$$

and define

$$D_n^{\mathbb{C}} := \{(b, a) \in \mathcal{W} : \gamma_n = 0\}. \tag{23}$$

In the sequel, we will omit the superscript and always write D_n for $D_n^{\mathbb{C}}$.

Similarly, we do this for the zeroes $(\dot{\lambda}_n)_{1\le n\le N-1}$ of $\dot{\Delta}_\lambda$. The λ_i's and $\dot{\lambda}_i$'s no longer depend continuously on $(b, a) \in \mathcal{M}^{\mathbb{C}}$. However, if we choose a small enough complex neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $\mathcal{M}^{\mathbb{C}}$, then for any $(b, a) \in \mathcal{W}$, the closed intervals $G_n \subseteq \mathbb{C}$ $(1 \le n \le N-1)$

defined by

$$G_n := \{(1-t)\lambda_{2n} + t\lambda_{2n+1} : 0 \le t \le 1\} \tag{24}$$

are pairwise disjoint, and hence, as in the real case, there exists a family of isolating neighborhoods $(U_n)_{1\le n\le N-1}$.

Lemma 2.3. There exists a neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $\mathcal{M}^{\mathbb{C}}$ such that for any $(b,a) \in \mathcal{W}$, there are neighborhoods U_n of G_n in $\mathbb{C}$ $(1 \le n \le N-1)$ which are pairwise disjoint. □

Remark 2.4. In the sequel, we will have to shrink the complex neighborhood $\mathcal{W}$ several times, but continue to denote it by the same letter. □

Contours Γ_n. For any $(b,a) \in \mathcal{W}$ and any $1 \le n \le N-1$, we denote by Γ_n a circuit in U_n around G_n with counterclockwise orientation.

Isospectral set. For $(b,a) \in \mathcal{M}$, the set $\mathrm{Iso}(b,a)$ of all elements $(b',a') \in \mathcal{M}$ so that $Q(b',a')$ has the same spectrum as $Q(b,a)$ is described with the help of the Dirichlet eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_{N-1}$ of equation (11) defined by

$$y_1(N+1,\mu_n) = 0. \tag{25}$$

They coincide with the eigenvalues of the $(N-1)\times(N-1)$-matrix $L_2 = L_2(b,a)$ given by

$$\begin{pmatrix} b_2 & a_2 & 0 & \dots & 0 \\ a_2 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ 0 & \dots & 0 & a_{N-1} & b_N \end{pmatrix}.$$

In the sequel, we will also refer to $\mu_1, \dots, \mu_{N-1}$ as the Dirichlet eigenvalues of $L(b,a)$. Evaluating the Wronskian identity (13) at $\lambda = \mu_n$ one sees that μ_n lies in the closure of the nth spectral gap. More precisely, substituting $y_1(N+1,\mu_n) = 0$ in identity (13) with $\lambda = \mu_n$ yields

$$y_1(N,\mu_n)y_2(N+1,\mu_n) = 1. \tag{26}$$

Hence the value of the discriminant at μ_n is given by

$$\Delta(\mu_n) = y_2(N+1,\mu_n) + \frac{1}{y_2(N+1,\mu_n)} \tag{27}$$

and $|\Delta(\mu_n)| \geq 2$. By Lemma 2.6, given the point (b, a) with $b_1 = \ldots = b_N = \beta$ and $a_1 = \ldots = a_N = \alpha$, one has $\lambda_{2n} = \lambda_{2n+1}$ and hence $\mu_n = \lambda_{2n}$ for any $1 \leq n \leq N-1$. It then follows from a straightforward deformation argument that $\lambda_{2n} \leq \mu_n \leq \lambda_{2n+1}$ everywhere in the real space $\mathcal{M}$.

Conversely, according to van Moerbeke [16], given any (real) Jacobi matrix Q with spectrum $\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \lambda_5 < \cdots \lambda_{2N-2} \leq \lambda_{2N-1} < \lambda_{2N}$ and any sequence $(\mu_n)_{1\leq n\leq N-1}$ with $\lambda_{2n} \leq \mu_n \leq \lambda_{2n+1}$ for $n = 1, \ldots, N-1$, there are exactly 2^r N-periodic Jacobi matrices Q with spectrum $(\lambda_n)_{1\leq n\leq 2N}$ and Dirichlet spectrum $(\mu_n)_{1\leq n\leq N-1}$, where r is the number of n's with $\lambda_{2n} < \mu_n < \lambda_{2n+1}$.

In the case where $(b, a) \in \mathcal{M}^{\mathbb{C}}$, we continue to define the Dirichlet eigenvalues $(\mu_n)_{1\leq n\leq N-1}$ by equation (25), and we list them in lexicographic ordering $\mu_1 \prec \mu_2 \prec \cdots \prec \mu_{N-1}$. Then the μ_i's no longer depend continuously on $(b, a) \in \mathcal{M}^{\mathbb{C}}$. However, if we choose the complex neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $\mathcal{M}^{\mathbb{C}}$ of Lemma 2.3 small enough, then for any $(b, a) \in \mathcal{W}$ and $1 \leq n \leq N-1$, μ_n is contained in the neighborhood U_n of G_n (but not necessarily in G_n itself).

Riemann surface $\Sigma_{b,a}$. Denote by $\Sigma_{b,a}$ the Riemann surface obtained as the compactification of the affine curve $\mathcal{C}_{b,a}$ defined by

$$\{(\lambda, z) \in \mathbb{C}^2 : z^2 = \Delta^2(\lambda, b, a) - 4\}. \tag{28}$$

Note that $\mathcal{C}_{b,a}$ and $\Sigma_{b,a}$ are spectral invariants. (Strictly speaking, $\Sigma_{b,a}$ is a Riemann surface only if the spectrum of $Q(b, a)$ is simple—see e.g. Appendix A in [17] for details in this case. If the spectrum of $Q(b, a)$ is *not* simple, $\Sigma(b, a)$ becomes a Riemann surface after doubling the multiple eigenvalues—see e.g. Section 2 of [13].)

Dirichlet divisors. To the Dirichlet eigenvalue μ_n $(1 \leq n \leq N-1)$ we associate the point μ_n^* on the surface $\Sigma_{b,a}$,

$$\mu_n^* := \left(\mu_n, \sqrt[*]{\Delta_{\mu_n}^2 - 4}\right) \text{ with } \sqrt[*]{\Delta_{\mu_n}^2 - 4} := y_1(N, \mu_n) - y_2(N+1, \mu_n), \tag{29}$$

where we used that, in view of equation (26) and the Wronskian identiy 14,

$$\Delta_{\mu_n}^2 - 4 = (y_1(N, \mu_n) - y_2(N+1, \mu_n))^2.$$

Standard root. The standard root or s-root for short, $\sqrt[s]{1-\lambda^2}$, is defined for $\lambda \in \mathbb{C} \setminus [-1,1]$ by

$$\sqrt[s]{1-\lambda^2} := i\lambda \sqrt[+]{1-\lambda^{-2}}. \tag{30}$$

More generally, we define for $\lambda \in \mathbb{C} \setminus \{ta + (1-t)b \,|\, 0 \leq t \leq 1\}$ the s-root of a radicand of the form $(b-\lambda)(\lambda-a)$ with $a \prec b, a \neq b$ by

$$\sqrt[s]{(b-\lambda)(\lambda-a)} := \frac{\gamma}{2} \sqrt[s]{1-w^2}, \tag{31}$$

where $\gamma := b - a$, $\tau := \frac{b+a}{2}$, and $w := \frac{\lambda-\tau}{\gamma/2}$.

Canonical sheet and canonical root. For $(b,a) \in \mathcal{M}$, the canonical sheet of $\Sigma_{b,a}$ is given by the set of points $(\lambda, \sqrt[c]{\Delta_\lambda^2 - 4})$ in $\mathcal{C}_{b,a}$, where the c-root $\sqrt[c]{\Delta_\lambda^2 - 4}$ is defined on $\mathbb{C} \setminus \bigcup_{n=0}^{N} (\lambda_{2n}, \lambda_{2n+1})$ (where $\lambda_0 := -\infty$ and $\lambda_{2N+1} := \infty$) and determined by the sign condition

$$-i\sqrt[c]{\Delta_\lambda^2 - 4} > 0 \quad \text{for} \quad \lambda_{2N-1} < \lambda < \lambda_{2N}. \tag{32}$$

As a consequence, one has for any $1 \leq n \leq N$

$$\operatorname{sign} \sqrt[c]{\Delta_{\lambda-i0}^2 - 4} = (-1)^{N+n-1} \quad \text{for} \quad \lambda_{2n} < \lambda < \lambda_{2n+1}. \tag{33}$$

The definition of the canonical sheet and the c-root can be extended to the neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $\mathcal{M}^{\mathbb{C}}$ of Lemma 2.3.

The s-root and the c-root will be used together in the following way: By the product representations (20) and (18) of $\dot{\Delta}_\lambda$ and $\Delta_\lambda^2 - 4$, respectively, one sees that for any (b,a) in $\mathcal{W} \setminus D_n$ with $1 \leq n \leq N-1$,

$$\frac{\dot{\Delta}_\lambda}{\sqrt[c]{\Delta_\lambda^2 - 4}} = \frac{N(\lambda - \dot{\lambda}_n)}{\sqrt[s]{(\lambda_{2n+1} - \lambda)(\lambda - \lambda_{2n})}} \chi_n(\lambda) \quad \forall \lambda \in \Gamma_n, \tag{34}$$

where

$$\chi_n(\lambda) = \frac{(-1)^{N+n-1}}{\sqrt[+]{(\lambda-\lambda_1)(\lambda_{2N}-\lambda)}} \prod_{m \neq n} \frac{\lambda - \dot{\lambda}_m}{\sqrt[+]{(\lambda - \lambda_{2m+1})(\lambda - \lambda_{2m})}}. \tag{35}$$

Note that the principal branches of the square roots in equation (35) are well defined for λ near G_n and that the function χ_n is analytic and nonvanishing on U_n. In addition, for (b, a) real, χ_n is non-negative on the interval $(\lambda_{2n}, \lambda_{2n+1})$.

Abelian differentials. Let $(b, a) \in \mathcal{M}$ and $1 \le n \le N-1$. Then there exists a unique polynomial $\psi_n(\lambda)$ of degree at most $N-2$ such that for any $1 \le k \le N-1$,

$$\frac{1}{2\pi} \int_{c_k} \frac{\psi_n(\lambda)}{\sqrt{\Delta_\lambda^2 - 4}} \, d\lambda = \delta_{kn}. \tag{36}$$

Here, for any $1 \le k \le N-1$, c_k denotes the lift of the contour Γ_k to the canonical sheet of $\Sigma_{b,a}$. For any $k \neq n$ with $\lambda_{2k} \neq \lambda_{2k+1}$, it follows from equation (36) that

$$\frac{1}{\pi} \int_{\lambda_{2k}}^{\lambda_{2k+1}} \frac{\psi_n(\lambda)}{\sqrt[+]{\Delta_\lambda^2 - 4}} \, d\lambda = 0. \tag{37}$$

Hence in every gap $(\lambda_{2k}, \lambda_{2k+1})$ with $k \neq n$, the polynomial ψ_n has a zero which we denote by σ_k^n. If $\lambda_{2k} = \lambda_{2k+1}$, then it follows from equation (36) and Cauchy's theorem that $\sigma_k^n = \lambda_{2k} = \lambda_{2k+1}$. As $\psi_n(\lambda)$ is a polynomial of degree at most $N-2$, one has

$$\psi_n(\lambda) = M_n \prod_{\substack{1 \le k \le N-1 \\ k \neq n}} \left(\lambda - \sigma_k^n\right), \tag{38}$$

where $M_n \equiv M_n(b, a) \neq 0$.

In a straightforward way, one can prove that there exists a neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $\mathcal{M}_{\mathbb{C}}$, so that for any $(b, a) \in \mathcal{W}$ and any $1 \le n \le N-1$, there is a unique polynomial $\psi_n(\lambda)$ of degree at most $N-2$ satisfying equation (36) for any $1 \le k \le N-1$ as well as the product representation (38), and so that the zeroes are analytic functions on $\mathcal{W}$.

Lemma 2.5. Let $1 \le n \le N-1$ be fixed. Then the zeroes $(\dot\lambda_k)_{1 \le k \le N-1}$ of $\dot\Delta(\lambda)$ and $(\sigma_k^n)_{1 \le k \le N-1, k \neq n}$ of $\psi_n(\lambda)$ satisfy the estimates

$$\dot\lambda_k - \tau_k = O\left(\gamma_k^2\right), \tag{39}$$

$$\sigma_k^n - \tau_k = O\left(\gamma_k^2\right) \tag{40}$$

near any given point $(b, a) \in \mathcal{W}$, where $\tau_k = \frac{1}{2}(\lambda_{2k+1} + \lambda_{2k})$. □

Proof. To verify equation (39), write $\Delta_\lambda^2 - 4$ in the form

$$\Delta_\lambda^2 - 4 = (\lambda - \lambda_{2n})(\lambda_{2n+1} - \lambda)\, p_n(\lambda), \tag{41}$$

where p_n is a polynomial which does not vanish for $\lambda \in U_n$. Then equation (39) follows by differentiating equation (41) with respect to λ at $\dot\lambda_n$.

Fix $1 \le k, n \le N-1$ with $k \ne n$. In the first step, we prove that $\sigma_k^n - \tau_k = O(\gamma_k)$ near any given point $(b,a) \in \mathcal{W}$. If $\gamma_k = 0$, then $\sigma_k^n = \tau_k$, and equation (40) is clearly fulfilled. Hence, we assume in the sequel that $\gamma_k \ne 0$. By the product formulas (38) and (18) for $\psi_n(\lambda)$ and $\Delta_\lambda^2 - 4$, respectively, we obtain, for λ near G_k,

$$\frac{\psi_n(\lambda)}{\sqrt[c]{\Delta_\lambda^2 - 4}} = \frac{\lambda - \sigma_k^n}{\sqrt[s]{(\lambda_{2k+1} - \lambda)(\lambda - \lambda_{2k})}} \zeta_k^n(\lambda), \tag{42}$$

where

$$\zeta_k^n(\lambda) = \frac{M_n'(b,a)}{\left(\lambda - \sigma_n^n\right)\sqrt[+]{(\lambda - \lambda_1)(\lambda_{2N} - \lambda)}} \prod_{m \ne k} \frac{\lambda - \sigma_m^n}{\sqrt[+]{(\lambda_{2m+1} - \lambda)(\lambda_{2m} - \lambda)}}, \tag{43}$$

with $\sigma_n^n := \tau_n$ and $M_n'(b,a) \ne 0$. The function ζ_k^n is analytic and nonvanishing in U_k. Substituting equation (42) into equation (36), one gets

$$\frac{1}{2\pi} \int_{\Gamma_k} \frac{\lambda - \sigma_k^n}{\sqrt[s]{(\lambda_{2k+1} - \lambda)(\lambda - \lambda_{2k})}} \zeta_k^n(\lambda) d\lambda = 0. \tag{44}$$

We now drop the superscript n for the remainder of this proof and write ζ_k as $\zeta_k(\lambda) = \xi_k + (\zeta_k(\lambda) - \xi_k)$ with $\xi_k := \zeta_k(\tau_k) \ne 0$. Note that

$$\frac{1}{2\pi} \int_{\Gamma_k} \frac{\lambda - \sigma_k}{\sqrt[s]{(\lambda_{2k+1} - \lambda)(\lambda - \lambda_{2k})}} \, d\lambda = \tau_k - \sigma_k,$$

and hence equation (44) becomes

$$(\sigma_k - \tau_k)\xi_k = \frac{1}{2\pi} \int_{\Gamma_k} \frac{(\lambda - \sigma_k)(\zeta_k(\lambda) - \xi_k)}{\sqrt[s]{(\lambda_{2k+1} - \lambda)(\lambda - \lambda_{2k})}} \, d\lambda. \tag{45}$$

To estimate the integral on the right-hand side of equation (45), note that

$$\left| \frac{1}{2\pi} \int_{\Gamma_k} \frac{f(\lambda)}{\sqrt[s]{(\lambda_{2k+1} - \lambda)(\lambda - \lambda_{2k})}} \, d\lambda \right| \leq \max_{\lambda \in G_k} |f(\lambda)| \tag{46}$$

for an arbitrary function f analytic on U_k. We want to apply equation (46) for $f(\lambda) = (\lambda - \sigma_k)(\zeta_k(\lambda) - \xi_k)$. Note that for $\lambda \in G_k$,

$$|\zeta_k(\lambda) - \xi_k| = |\zeta_k(\lambda) - \zeta_k(\tau_k)| \leq M|\gamma_k|,$$

where $M = \sup \bigcup_{1 \leq k \leq N-1} \{|\zeta_k(\lambda)| : \lambda \in G_k\}$. Hence equation (46) leads to

$$|\sigma_k - \tau_k||\xi_k| = \sup_{\lambda \in G_k} |\lambda - \sigma_k| \, O(\gamma_k).$$

Dividing by $|\xi_k| \neq 0$, we get

$$|\sigma_k - \tau_k| = \sup_{\lambda \in G_k} |\lambda - \sigma_k| \, O(\gamma_k) \tag{47}$$

and in particular, $|\sigma_k - \tau_k| = O(\gamma_k)$.

In the second step, we now improve estimate (47). Note that

$$\sup_{\lambda \in G_k} |\lambda - \sigma_k| \leq |\sigma_k - \tau_k| + \sup_{\lambda \in G_k} |\lambda - \tau_k| = O(\gamma_k). \tag{48}$$

Combining equations (47) and (48), we obtain the claimed estimate (40). ■

For later use, we compute the spectra of $Q(b, a)$ and $L_2(b, a)$ in the special case $(b, a) = (\beta 1_N, \alpha 1_N)$ with $\beta \in \mathbb{R}$ and $\alpha > 0$. Here 1_N denotes the vector $(1, \ldots, 1) \in \mathbb{R}^N$. These points are the equilibrium points (of the restrictions) of the Toda Hamiltonian vector field (to the symplectic leaves $\mathcal{M}_{\beta,\alpha}$). We compute the spectrum $(\lambda_j)_{1 \leq j \leq 2N}$ of the matrix $Q(\beta 1_N, \alpha 1_N)$ and the Dirichlet eigenvalues $(\mu_k)_{1 \leq k \leq N-1}$ of $L = L(\beta 1_N, \alpha 1_N)$ together with a normalized eigenvector $g_l = (g_l(j))_{1 \leq j \leq N}$ of μ_l, i.e. $L g_l = \mu_l g_l$, $g_l(1) = 0$, and a vector $h_l = (h_l(j))_{1 \leq j \leq N}$, which is the normalized solution of $Ly = \mu_l y$ orthogonal to g_l satisfying $W(h_l, g_l)(N) > 0$.

Lemma 2.6. The spectrum $(\lambda_j)_{1\le j\le 2N}$ of $Q(\beta 1_N, \alpha 1_N)$ and the Dirichlet eigenvalues $(\mu_l)_{1\le l\le N-1}$ of $L(\beta 1_N, \alpha 1_N)$ are given by

$$\lambda_1 = \beta - 2\alpha,$$
$$\lambda_{2l} = \lambda_{2l+1} = \mu_l = \beta - 2\alpha \cos\frac{l\pi}{N} \quad (1 \le l \le N-1),$$
$$\lambda_{2N} = \beta + 2\alpha.$$

In particular, all spectral gaps of $Q(\beta 1_N, \alpha 1_N)$ are collapsed. For any $1 \le l \le N-1$, the vectors g_l and h_l defined by

$$g_l(j) = (-1)^{j+1}\sqrt{\frac{2}{N}}\sin\frac{(j-1)l\pi}{N} \quad (1 \le j \le N), \tag{49}$$
$$h_l(j) = (-1)^{j}\sqrt{\frac{2}{N}}\cos\frac{(j-1)l\pi}{N} \quad (1 \le j \le N) \tag{50}$$

satisfy $Ly = \mu_l y$ and the normalization conditions

$$\sum_{j=1}^{N} g_l(j)^2 = \sum_{j=1}^{N} h_l(j)^2 = 1, \quad g_l(0) > 0, \quad g_l(1) = 0;$$
$$W(h_l, g_l)(N) > 0, \quad \langle h_l, g_l\rangle_{\mathbb{R}^N} = 0.$$

□

Remark 2.7. For $(b, a) = (\beta 1_N, \alpha 1_N)$, the fundamental solutions y_1 and y_2 are given by

$$y_1(j,\lambda) = -\frac{\sin(\rho(j-1))}{\sin\rho} \quad (j \in \mathbb{Z}), \tag{51}$$
$$y_2(j,\lambda) = \frac{\sin(\rho j)}{\sin\rho} \quad (j \in \mathbb{Z}), \tag{52}$$

where $\pi < \rho < 2\pi$ is determined by $\cos\rho = \frac{\lambda-\beta}{2\alpha}$. □

Proof. For any $\lambda \in \mathbb{R}$, the difference equation (11) for $(\beta 1_N, \alpha 1_N)$ reads

$$(R_{\beta,\alpha} y)(k) := \beta y(k) + \alpha y(k-1) + \alpha y(k+1) = \lambda y(k) \tag{53}$$

and can be written as

$$y(k-1) + y(k+1) = \frac{\lambda-\beta}{\alpha} y(k). \tag{54}$$

Since we are looking for periodic solutions of equation (54), we make the ansatz $y(k) = e^{\pm i\rho k}$. This leads to the characteristic equation

$$2\cos\rho \equiv e^{i\rho} + e^{-i\rho} = \frac{\lambda - \beta}{\alpha}.$$

For the solution to be $2N$-periodic, it is required that $\rho \in \frac{\pi}{N}\mathbb{Z}$. To put the eigenvalues in ascending order, introduce $\rho_l = (1 + \frac{l}{N})\pi$ with $0 \le l \le N$. Then for any $1 \le j \le 2N$, there exists $0 \le l \le N$ such that

$$\lambda_j = \beta + 2\alpha\cos\rho_l = \beta - 2\alpha\cos\frac{l\pi}{N}.$$

Note that for $l = 0$, $\lambda_1 = \beta - 2\alpha$ is an eigenvalue of $Q(\beta 1_N, \alpha 1_N)$ with eigenvector $y(k) = e^{i\pi k} = (-1)^k$. Similarly, for $l = N$, $\lambda_{2N} = \beta + 2\alpha$ is an eigenvalue with eigenvector $y(k) \equiv 1$. For the eigenvalue $\lambda_{2l} = \beta - 2\alpha\cos\frac{l\pi}{N}$ $(1 \le l \le N-1)$,

$$y_\pm(k) = e^{\pm i\rho_l k}$$

are two linearly independent eigenvectors. As there are $2N$ eigenvalues altogether, λ_{2l} is double for any $1 \le l \le N-1$, and λ_1 and λ_{2N} are both simple. It follows that all $N-1$ gaps are collapsed and hence $\mu_l = \lambda_{2l}$ for all $1 \le l \le N-1$.

Turning to the computation of g_k and h_k, one easily verifies that for any real number $\lambda \ne \pm 2\alpha + \beta$, the fundamental solution $y_1(\cdot, \lambda)$ of equation (54) with $y_1(0, \lambda) = 1$ and $y_1(1, \lambda) = 0$ is given by

$$y_1(j, \lambda) = -\frac{\sin\big(\rho(j-1)\big)}{\sin\rho} \qquad (j \in \mathbb{Z}),$$

where $\pi < \rho < 2\pi$ is determined by $\cos\rho = \frac{\lambda-\beta}{2\alpha}$, thus proving equation (51). In the same way, one verifies equation (52). For $\lambda = \mu_l = \beta - 2\alpha\cos\frac{l\pi}{N}$, we then get

$$\sin\big(\rho_l(j-1)\big) = \sin\left(\left(1 + \frac{l}{N}\right)\pi(j-1)\right) = (-1)^{j+1}\sin\frac{(j-1)l\pi}{N}.$$

In particular, $\sin\big(\rho_l(j-1)\big) = 0$ for $j = 1$ and $j = N+1$. As

$$\sum_{j=1}^{N}\sin^2\frac{(j-1)l\pi}{N} = \sum_{j=1}^{N}\cos^2\frac{(j-1)l\pi}{N},$$

and these two sums add up to N, one sees that

$$\sum_{j=1}^{N} \sin^2 \frac{(j-1)l\pi}{N} = \frac{N}{2}, \tag{55}$$

yielding the claimed formula (49) for g_l.

By the same argument, one shows that $\tilde{h}_l$ given by $(-1)^j \sqrt{\frac{2}{N}} \cos \frac{(j-1)l\pi}{N}$ (i.e. the right side of equation (50)) satisfies $R_{\beta,\alpha}\tilde{h}_l = \mu_l \tilde{h}_l$ and the normalization condition $\sum_{j=1}^{N} \tilde{h}_l(j)^2 = 1$. Using standard trigonometric identities one verifies that

$$\langle g_l, \tilde{h}_l \rangle = \sum_{j=1}^{N} g_l(j)\tilde{h}_l(j) = 0$$

and $W(\tilde{h}_l, g_l)(N)$ can be computed to be

$$\tilde{h}_l(N)g_l(N+1) - \tilde{h}_l(N+1)g_l(N) = -\tilde{h}_l(N+1)g_l(N) = -\tilde{h}_l(1)g_l(0) > 0.$$

Hence $\tilde{h}_l$ is indeed the eigenvector with the required normalization, i.e. $h_l = \tilde{h}_l$, thus proving equation (50). ■

3 Action Variables

In the next two sections, we define the candidates for action–angle variables on the phase space $\mathcal{M}$ and investigate some of their properties. In this section, we introduce globally defined action variables $(I_n)_{1 \le n \le N-1}$ as proposed by Flaschka–McLaughlin [5].

Definition 3.1. Let $(b, a) \in \mathcal{M}$. For $1 \le n \le N-1$,

$$I_n := \frac{1}{2\pi} \int_{\Gamma_n} \lambda \frac{\dot{\Delta}_\lambda}{\sqrt[c]{\Delta_\lambda^2 - 4}} \, d\lambda, \tag{56}$$

where $\dot{\Delta}_\lambda = \frac{d}{d\lambda}\Delta_\lambda$ is the λ-derivative of the discriminant $\Delta_\lambda = \Delta(\lambda, b, a)$ and the contour Γ_n and the canonical root $\sqrt[c]{\cdot}$ are given as in Section 2. □

Remark 3.2. The contours Γ_n can be chosen locally independently of (b, a). In view of the fact that Δ_λ is a spectral invariant of $L(b, a)$, the actions I_n are entirely determined by the spectrum of $L(b, a)$. In particular, $(I_n)_{1 \le n \le N-1}$ are constants of motion, since by Proposition 2.1, the Toda flow is isospectral. □

Remark 3.3. The variables $(I_n)_{1 \le n \le N-1}$ can also be represented as integrals on the Riemann surface $\Sigma_{b,a}$. If c_n denotes the lift of Γ_n to the canonical sheet of $\Sigma_{b,a}$, equation (56) becomes

$$I_n = \frac{1}{2\pi} \int_{c_n} \lambda \frac{\dot{\Delta}_\lambda}{\sqrt{\Delta_\lambda^2 - 4}} d\lambda \quad (1 \le n \le N-1). \tag{57}$$

□

From definition (56), the following result can be deduced.

Proposition 3.4. On the real space $\mathcal{M}$, each function I_n is real, non-negative, and it vanishes if $\gamma_n = 0$. □

Proof. Since

$$\int_{\Gamma_n} \frac{\dot{\Delta}_\lambda}{\sqrt[c]{\Delta_\lambda^2 - 4}} d\lambda = 0,$$

it follows that

$$I_n = \frac{1}{2\pi} \int_{\Gamma_n} (\lambda - \dot{\lambda}_n) \frac{\dot{\Delta}_\lambda}{\sqrt[c]{\Delta_\lambda^2 - 4}} d\lambda. \tag{58}$$

By shrinking the contour of integration to the real interval, we get

$$I_n = \frac{1}{\pi} \int_{\lambda_{2n}}^{\lambda_{2n+1}} (-1)^{N+n-1} (\lambda - \dot{\lambda}_n) \frac{\dot{\Delta}_\lambda}{\sqrt[+]{\Delta_\lambda^2 - 4}} d\lambda$$

by taking into account definition (32) of the c-root. Since $\operatorname{sign}(\lambda - \dot{\lambda}_n)\dot{\Delta}_\lambda = (-1)^{N+n-1}$ on $[\lambda_{2n}, \lambda_{2n+1}] \setminus \{\dot{\lambda}_n\}$, the integrand is real and non-negative, hence I_n is real and non-negative on $\mathcal{M}$, as claimed.

If $\gamma_n = 0$, then $\lambda_{2n} = \lambda_{2n+1}$. Hence $\dot{\lambda}_n = \lambda_{2n} = \lambda_{2n+1} = \tau_n$ and

$$\lambda - \dot{\lambda}_n = i \sqrt[s]{(\lambda_{2n+1} - \lambda)(\lambda - \lambda_{2n})}.$$

Therefore the integrand in equation (56) is holomorphic in the interior of the contour Γ_n, and by Cauchy's theorem the integral in equation (56) vanishes. ■

The action variables $(I_n)_{1\le n\le N-1}$ can be extended in a straightforward way to a complex neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $\mathcal{M}^{\mathbb{C}}$.

Theorem 3.5. There exists a complex neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $\mathcal{M}^{\mathbb{C}}$ such that for all $1 \le n \le N-1$, the functions I_n defined by equation (56) extend analytically to $\mathcal{W}$, $I_n : \mathcal{W} \to \mathbb{C}$. □

Proof. Let $\mathcal{W}$ denote a neighborhood of $\mathcal{M}$ in $\mathcal{M}^{\mathbb{C}}$ of Lemma 2.3 and define for any $1 \le n \le N-1$ the functions I_n on $\mathcal{W}$ by formula (56). Let $(b, a) \in \mathcal{W}$ be given. Then there exists a neighborhood $\mathcal{W}_{b,a}$ of (b, a) in $\mathcal{W}$, so that the integration contours Γ_n in formula (56) can be chosen to be the same for any element in $\mathcal{W}_{b,a}$ and $\dot{\Delta}_\lambda/\sqrt[c]{\Delta_\lambda^2 - 4}$ is analytic on $B_\varepsilon(\Gamma_n) \times \mathcal{W}_{b,a}$, where $B_\varepsilon(\Gamma_n) := \{\lambda \in \mathbb{C}|\, \mathrm{dist}\,(\lambda, \Gamma_n) < \varepsilon\}$ is the ε-neighborhood of Γ_n with ε sufficiently small. This shows that I_n is analytic on $\mathcal{W}$. ■

Proposition 3.6. There exists a complex neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $\mathcal{M}^{\mathbb{C}}$ such that for any $1 \le n \le N-1$, the quotient I_n/γ_n^2 extends analytically from $\mathcal{M} \setminus D_n$ to all of $\mathcal{W}$ and has strictly positive real part on $\mathcal{W}$. As a consequence, $\xi_n = \sqrt[+]{2I_n/\gamma_n^2}$ is an analytic and nonvanishing function on $\mathcal{W}$, where $\sqrt[+]{\cdot}$ is the principal branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$. □

Proof. Let $\mathcal{W}$ be the complex neighborhood of Theorem 3.5. Substituting equation (34) into equation (58) leads to the following identity on $\mathcal{W} \setminus D_n$:

$$I_n = \frac{N}{2\pi} \int_{\Gamma_n} \frac{(\lambda - \dot{\lambda}_n)^2}{\sqrt[s]{(\lambda_{2n+1} - \lambda)(\lambda - \lambda_{2n})}} \chi_n(\lambda) d\lambda,$$

where χ_n is given by equation (35). Upon the substitution $\lambda(\zeta) = \tau_n + \frac{\gamma_n}{2}\zeta$, with $\tau_n = \frac{1}{2}(\lambda_{2n} + \lambda_{2n+1})$ and $\delta_n = \frac{2(\dot{\lambda}_n - \tau_n)}{\gamma_n}$, one then obtains

$$\frac{2I_n}{\gamma_n^2} = \frac{N}{4\pi} \int_{\Gamma_n'} \frac{(\zeta - \delta_n)^2}{\sqrt[s]{1 - \zeta^2}} \chi_n\left(\tau_n + \frac{\gamma_n}{2}\zeta\right) d\zeta, \tag{59}$$

where Γ_n' is the pullback of Γ_n under the substitution $\lambda = \lambda(\zeta)$, i.e. a circuit in $\mathbb{C}$ around $[-1,1]$. By equation (39), $\dot{\lambda}_n - \tau_n = O(\gamma_n^2)$, and hence $\delta_n \to 0$ as $\gamma_n \to 0$. We conclude that

$$\lim_{\gamma_n\to 0} \frac{2I_n}{\gamma_n^2} = \frac{N}{4\pi}\int_{\Gamma_n'} \chi_n(\tau_n)\frac{\zeta^2\,d\zeta}{\sqrt[s]{1-\zeta^2}} = \chi_n(\tau_n)\frac{N}{2\pi}\int_{-1}^{1}\frac{t^2\,dt}{\sqrt[+]{1-t^2}} = \frac{N}{4}\chi_n(\tau_n).$$

By defining $\frac{2I_n}{\gamma_n^2}$ by $\frac{N}{4}\chi_n(\tau_n)$ on $\mathcal{W}\cap D_n$, it follows that $\frac{2I_n}{\gamma_n^2}$ is a continuous function on all of $\mathcal{W}$. This extended function is analytic on $\mathcal{W}\setminus D_n$ as is its restriction to $\mathcal{W}\cap D_n$. By Theorem A.6 in [12], it then follows that $\frac{2I_n}{\gamma_n^2}$ is analytic on all of $\mathcal{W}$.

By Lemma 3.7, the quotient I_n/γ_n^2 can be bounded away from zero on $\mathcal{M}$, $\frac{I_n}{\gamma_n^2} \ge \frac{1}{3\pi(\lambda_{2N}-\lambda_1)}$. By shrinking $\mathcal{W}$, if necessary, it then follows that for any $1 \le n \le N-1$, the real part of I_n/γ_n^2 is positive and never vanishes on $\mathcal{W}$. Hence the principal branch of the square root of $2I_n/\gamma_n^2$ is well defined on $\mathcal{W}$ and ξ_n has the claimed properties. ■

To show that $\sqrt[+]{\frac{2I_n}{\gamma_n^2}}$ is well defined on $\mathcal{W}$, we used in the proof of Proposition 3.6 the following auxiliary result, which we prove in Appendix A.

Lemma 3.7. For any $(b,a)\in\mathcal{M}$ and any $1 \le n \le N-1$,

$$\gamma_n^2 \le 3\pi(\lambda_{2N}-\lambda_1)I_n. \tag{60}$$

□

From definition (56), Proposition 3.4, and estimate (60), one obtains the following corollary.

Corollary 3.8. For any $(b,a)\in\mathcal{M}$ and any $1 \le n \le N-1$,

$$I_n = 0 \quad \text{if and only if} \quad \gamma_n = 0.$$

□

Actually, Lemma 3.7 can be improved. We finish this section with an a priori estimate of the gap lengths γ_n in terms of the action variables and the value of the Casimir C_2 alone, which will be shown in Appendix B.

Theorem 3.9. For any $(b,a)\in\mathcal{M}_{\beta,\alpha}$ with $\beta\in\mathbb{R}$, $\alpha>0$ arbitrary,

$$\sum_{n=1}^{N-1}\gamma_n^2 \le 12\pi^2\alpha\left(\sum_{n=1}^{N-1}I_n\right) + 9\pi^2(N-1)\left(\sum_{n=1}^{N-1}I_n\right)^2. \tag{61}$$

□

4 Angle Variables

In this section, we define and study the angle coordinates $(\theta_n)_{1\le n\le N-1}$. Each θ_n is defined mod 2π on $\mathcal{W}\setminus D_n$, where $\mathcal{W}$ is a complex neighborhood of $\mathcal{M}$ in $\mathcal{M}^{\mathbb{C}}$ as in Lemma 2.3 and D_n is given by equation (23).

Definition 4.1. For any $1\le n\le N-1$, the function θ_n is defined for $(b,a)\in\mathcal{M}\setminus D_n$ by

$$\theta_n := \eta_n + \sum_{n\neq k=1}^{N-1} \beta_k^n \mod 2\pi, \tag{62}$$

where for $k\neq n$,

$$\beta_k^n = \int_{\lambda_{2k}}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta_\lambda^2-4}}\,d\lambda, \quad \eta_n = \int_{\lambda_{2n}}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta_\lambda^2-4}}\,d\lambda \ (\mathrm{mod}\ 2\pi), \tag{63}$$

and where for $1\le k\le N-1$, μ_k^* is the Dirichlet divisor defined in equation (29), and λ_{2k} is identified with the ramification point $(\lambda_{2k},0)$ on the Riemann surface $\Sigma_{b,a}$. The integration paths on $\Sigma_{b,a}$ in equation (63) are required to be admissible in the sense that their image under the projection $\pi:\Sigma_{b,a}\to\mathbb{C}$ on the first component stays inside the isolating neighborhoods U_k. □

Note that, in view of the normalization conditions (36) of ψ_n, the above restriction of the paths of integration in equation (63) implies that η_n and hence θ_n are well-defined mod 2π.

Theorem 4.2. Let $\mathcal{W}$ be the complex neighborhood of $\mathcal{M}$ in $\mathcal{M}^{\mathbb{C}}$ introduced in Lemma 2.3. Then for any $1\le n\le N-1$, the function $\theta_n:\mathcal{W}\setminus D_n\to\mathbb{C}\,(\mathrm{mod}\,\pi)$ is analytic. □

Remark 4.3. As the lexicographic ordering of the eigenvalues of $Q(b,a)$ is not continuous on $\mathcal{W}$, it follows that η_n and hence θ_n are only continuous mod π on $\mathcal{W}$. □

Proof of Theorem 4.2 To see that $\theta_n:\mathcal{W}\setminus D_n\to\mathbb{C}$ (mod π) is analytic, define for any $1\le k\le N-1$ the set

$$E_k := \{(b,a)\in\mathcal{M}^{\mathbb{C}} : \mu_k(b,a)\in\{\lambda_{2k}(b,a),\lambda_{2k+1}(b,a)\}\}.$$

Below, we show that for any $1\le k\le N-1$ with $k\neq n$, β_k^n is analytic on $\mathcal{W}\setminus(D_k\cup E_k)$, that its restrictions to $D_k\cap\mathcal{W}$ and $E_k\cap\mathcal{W}$ are weakly analytic and that it is continuous on $\mathcal{W}$. (Let E and F be complex Banach spaces, and let $U \subset E$ be open. The map $f : U \to F$ is *weakly analytic on* U, if for each $u \in U$, $h \in E$, and $L \in F^*$, the function $z \mapsto Lf(u+zh)$ is analytic in some neighborhood of the origin in $\mathbb{C}$.) Together with the fact that $E_k \cap \mathcal{W}$ and $D_k \cap \mathcal{W}$ are analytic subvarieties of $\mathcal{W}$, it then follows that β_k^n is analytic on $\mathcal{W}$—see Theorem A.6 in [12]. Similar results can be shown for $\beta_n^n = \eta_n \pmod{\pi}$ on $\mathcal{W} \setminus D_n$, and one concludes that $\theta_n \pmod{\pi}$ is analytic on $\mathcal{W} \setminus D_n$.

To prove that β_k^n, $k \neq n$, is analytic on $\mathcal{W} \setminus (D_k \cup E_k)$, note that since λ_{2k} is a simple eigenvalue on $\mathcal{W} \setminus D_k$, it is analytic there. Furthermore, μ_k^* is an analytic function on the (sufficiently small) neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $\mathcal{M}^{\mathbb{C}}$. On $\mathcal{W} \setminus (D_k \cup E_k)$, we can use the substitution $\lambda = \lambda_{2k} + z$ to get

$$\beta_k^n = \int_{\lambda_{2k}}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta_\lambda^2 - 4}} d\lambda = \int_0^{\mu_k^* - \lambda_{2k}} \frac{\psi_n(\lambda_{2k} + z)}{\sqrt{z}\sqrt{D(z)}} dz,$$

where $D(z) = \frac{\Delta^2(\lambda_{2k}+z)-4}{z}$ is analytic near $z = 0$ and $D(0) \neq 0$. Note that $D(z)$ does not vanish for z on an admissible integration path not going through λ_{2k+1}. Such a path exists, since (b, a) is in the complement of E_k. Furthermore, $\psi_n(\lambda_{2k} + z)$ and $D(z)$ are analytic in z near such a path and depend analytically on $(b, a) \in \mathcal{W} \setminus (D_k \cup E_k)$. Combining these arguments shows that β_k^n is analytic on $\mathcal{W} \setminus (D_k \cup E_k)$.

For $k \neq n$ with $\lambda_{2k} \neq \lambda_{2k+1}$, one has

$$\int_{\lambda_{2k}}^{\lambda_{2k+1}} \frac{\psi_n(\lambda)}{\sqrt{\Delta_\lambda^2 - 4}} d\lambda = 0. \tag{64}$$

As $\sigma_k^n = \lambda_{2k}$ if $\lambda_{2k} = \lambda_{2k+1}$, one sees that equation (64) continues to hold for $(b, a) \in E_k \cap W$ with $\lambda_{2k} = \lambda_{2k+1}$ and we have $\beta_k^n|_{E_k \cap \mathcal{W}} \equiv 0$. To prove the analyticity of $\beta_k^n|_{D_k \cap \mathcal{W}}$, consider the representation (42) of $\frac{\psi_n(\lambda)}{\sqrt{\Delta_\lambda^2 - 4}}$. For $(b, a) \in D_k \cap \mathcal{W}$, one has

$$\lambda_{2k} = \lambda_{2k+1} = \tau_k = \sigma_k^n,$$

which implies that the factor $\frac{\lambda - \sigma_k^n}{\sqrt[s]{(\lambda_{2k+1} - \lambda)(\lambda - \lambda_{2k})}}$ in equation (42) equals $\pm i$. Hence we can write

$$\beta_k^n = \int_{\lambda_{2k}}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta_\lambda^2 - 4}}\, d\lambda = \pm i \int_{\tau_k}^{\mu_k} \zeta_k^n(\lambda) d\lambda.$$

As μ_k is analytic on $\mathcal{W}$, it then follows that $\beta_k^n|_{D_k \cap \mathcal{W}}$ is analytic. To see that β_k^n is continuous on $\mathcal{W}$, one separately shows that β_k^n is continuous at points in $\mathcal{W} \setminus (D_k \cup E_k)$, $E_k \cap \mathcal{W} \setminus D_k$,

$D_k \cap \mathcal{W} \setminus E_k$, and $D_k \cap E_k \cap \mathcal{W}$, where for the proof of the continuity of β_k^n at points in $D_k \cap E_k \cap \mathcal{W}$, we use equation (42) and the estimate $\sigma_k^n - \tau_k = O(\gamma_k^2)$ of Lemma 2.5.

By equation (64), η_n vanishes mod π on $E_n \cap \mathcal{W} \setminus D_n$. Arguing in a similar way as for β_k^n, one then concludes that η_n (mod π) is analytic on $\mathcal{W} \setminus D_n$. ■

5 Gradients

In this section, we establish formulas of the gradients of I_n, θ_n $(1 \le n \le N-1)$ on $\mathcal{M}$ in terms of products of the fundamental solutions y_1 and y_2.

Consider the discriminant for a fixed value of λ as a function on $\mathcal{M}$,

$$\Delta_\lambda(b,a) = y_1(N) + y_2(N+1).$$

Then Δ_λ is a real analytic function on $\mathcal{M}$. To obtain a formula for the gradients of $y_1(N)$ and $y_2(N+1)$ with respect to b, differentiate $R_{b,a}\, y_i = \lambda y_i$ with respect to b in the direction $v \in \mathbb{R}^N$ to get

$$(R_{b,a} - \lambda)\langle \nabla_b y_i, v\rangle(k) = -v_k\, y_i(k). \tag{65}$$

Differentiating $R_{b,a}\, y_i = \lambda y_i$ with respect to a in the direction $u \in \mathbb{R}^N$ leads to

$$(R_{b,a} - \lambda)\langle \nabla_a y_i, u\rangle(k) = -u_{k-1}\, y_i(k-1) - u_k\, y_i(k+1). \tag{66}$$

Taking the sum of equations (65) and (66) yields

$$(R_{b,a} - \lambda)\left(\langle \nabla_b y_i, v\rangle + \langle \nabla_a y_i, u\rangle\right)(k) = -(R_{v,u}\, y_i)(k), \tag{67}$$

which we can rewrite as

$$(R_{b,a} - \lambda)\langle \nabla_{b,a} y_i, (v,u)\rangle(k) = -(R_{v,u}\, y_i)(k), \tag{68}$$

where $\langle\cdot,\cdot\rangle$ in equation (68) now denotes the standard scalar product in $\mathbb{R}^{2N}$, whereas in equations (65), (66), and (67), it is the one in $\mathbb{R}^N$. The inhomogeneous Jacobi difference equation (68) for the sequence $\langle \nabla_{b,a} y_i, (v,u)\rangle(k)$ can be integrated using the discrete analogue of the method of the variation of constants used for inhomogeneous differential equations. As $\langle \nabla_{b,a} y_i, (v,u)\rangle(0) = \langle \nabla_{b,a} y_i, (v,u)\rangle(1) = 0$, one obtains in this way for $m \ge 1$

$$\langle \nabla_{b,a}\, y_i, (v,u)\rangle(m) = -\left(\frac{y_2(m)}{a_N}\sum_{k=1}^{m} y_1(k)(R_{v,u}\, y_i)(k) - \frac{y_1(m)}{a_N}\sum_{k=1}^{m} y_2(k)(R_{v,u}\, y_i)(k)\right). \tag{69}$$

In the sequel, we will use equation (69) to derive various formulas for the gradients. The common feature among these formulas is that they involve products between the fundamental solutions y_1 and y_2 of equation (11). Whereas the gradients with respect to $b = (b_1, \dots, b_N)$ involve products computed by componentwise multiplication, the gradients with respect to $a = (a_1, \dots, a_N)$ involve products obtained by multiplying shifted components, reflecting the fact that the b_j are the diagonal elements of the symmetric matrix $L(b, a)$, whereas the a_j are the off-diagonal elements of $L(b, a)$.

To simplify notation for the formulas in this section, we define for sequences $(v(j)_{j\in\mathbb{Z}}), (w(j)_{j\in\mathbb{Z}}) \subseteq \mathbb{C}$ the N-vectors

$$v \cdot w := (v(j)w(j))_{1\leq j\leq N}, \tag{70}$$

$$v \cdot Sw := (v(j)w(j+1))_{1\leq j\leq N}, \tag{71}$$

where S denotes the shift operator of order 1. Combining equations (70) and (71), we define the $2N$-vector

$$v \cdot_{\mathbf{S}} w := (v \cdot w, v \cdot Sw + w \cdot Sv). \tag{72}$$

In case $v = w$, we also use the shorter notation

$$v^2 := v \cdot_{\mathbf{S}} v. \tag{73}$$

Written componentwise, $v \cdot_{\mathbf{S}} w$ is the $2N$-vector

$$(v \cdot_{\mathbf{S}} w)(j) = \begin{cases} v(j)w(j) & (1 \leq j \leq N) \\ v(j-N)w(j-N+1) + v(j-N+1)w(j-N) & (N < j \leq 2N). \end{cases}$$

Proposition 5.1. For any $(b, a) \in \mathcal{M}$, the gradient $\nabla_{b,a}\Delta_\lambda = (\nabla_b\Delta_\lambda, \nabla_a\Delta_\lambda)$ is given by

$$-a_N\nabla_b\Delta_\lambda = y_2(N)\, y_1 \cdot y_1 - y_1(N{+}1)\, y_2 \cdot y_2 + \big(y_2(N{+}1) - y_1(N)\big)\, y_1 \cdot y_2 \tag{74}$$

$$-a_N\nabla_a\Delta_\lambda = 2y_2(N)y_1 \cdot Sy_1 - 2y_1(N+1)y_2 \cdot Sy_2 + \big(y_2(N+1) - y_1(N)\big)\big(y_1 \cdot Sy_2 + y_2 \cdot Sy_1\big) \tag{75}$$

or in the notation introduced above

$$\nabla_{b,a}\Delta_\lambda = -\frac{1}{a_N}\left(y_2(N)y_1^2 - y_1(N+1)y_2^2 + \big(y_2(N+1) - y_1(N)\big)\, y_1 \cdot_{\mathbf{s}} y_2\right). \tag{76}$$

The gradients $\nabla_b\Delta_\lambda$ and $\nabla_a\Delta_\lambda$ admit the representations ($1 \le m \le N$),

$$\frac{\partial\Delta_\lambda}{\partial b_m} = -\frac{1}{a_m}\, y_2(N, \lambda, S^m b, S^m a), \tag{77}$$

$$\frac{\partial\Delta_\lambda}{\partial a_m} = -\left(\frac{1}{a_m}\, y_2(N{+}1, \lambda, S^m b, S^m a) + \frac{1}{a_{m+1}}\, y_2(N{-}1, \lambda, S^{m+1} b, S^{m+1} a)\right). \tag{78}$$

□

Proof. The claimed formula (76) follows from the definition of Δ_λ and formula (69). Indeed, evaluate equation (69) for $i = 1$ and $m = N$ to get

$$\begin{aligned}\langle\nabla_{b,a}\, y_1, (v,u)\rangle(N) = &-\frac{y_2(N)}{a_N}\sum_{k=1}^{N} y_1(k)\left(u_{k-1}\, y_1(k-1) + v_k\, y_1(k) + u_k\, y_1(k+1)\right)\\ &+\frac{y_1(N)}{a_N}\sum_{k=1}^{N} y_2(k)\left(u_{k-1}\, y_1(k-1) + v_k\, y_1(k) + u_k\, y_1(k+1)\right).\end{aligned} \tag{79}$$

In order to identify these two sums with $\langle y_1^2, (v,u)\rangle$ and $\langle y_1 \cdot_{\mathbf{s}} y_2, (v,u)\rangle$, respectively, note that

$$\sum_{k=1}^{N} u_{k-1}\, y_1(k) y_1(k-1) = \sum_{k=1}^{N} u_k\, y_1(k) y_1(k+1) + u_N T_1,$$

where

$$T_1 := y_1(0)y_1(1) - y_1(N)y_1(N+1).$$

For the second sum in equation (79), we get an expression of the same type with a similar correction term

$$T_2 := y_1(0)y_2(1) - y_1(N)y_2(N+1).$$

Taking into account the initial conditions of the fundamental solutions and the Wronskian identity (13), one sees that $y_2(N)T_1 - y_1(N)T_2$ vanishes. Hence we have the formula

$$\langle\nabla_{b,a}\, y_1, (v,u)\rangle(N) = -\frac{1}{a_N}\left(y_2(N)\langle y_1^2, (v,u)\rangle - y_1(N)\langle y_1 \cdot_{\mathbf{s}} y_2, (v,u)\rangle\right). \tag{80}$$

Similarly, evaluating formula (69) for $i = 2$ and $m = N+1$ leads to

$$\langle \nabla_{b,a} y_2, (v,u)\rangle(N+1) = -\frac{1}{a_N}\left(y_2(N+1)\langle y_1 \cdot_{\mathbf{S}} y_2, (v,u)\rangle - y_1(N+1)\langle y_2^2, (v,u)\rangle\right). \tag{81}$$

Here we used that the value of the right side of equation (69) does not change when we omit the term for $k = m = N+1$ in both sums.

It remains to prove the two formulas (77) and (78). We first note that

$$y_2(n,\lambda,S^m b, S^m a) = \frac{a_m}{a_N}(y_2(n+m,\lambda,b,a)y_1(m,\lambda,b,a) - y_1(n+m,\lambda,b,a)y_2(m,\lambda,b,a)), \tag{82}$$

since both sides of equation (82) are solutions of $R_{S^m b, S^m a}\, y = \lambda y$ (for fixed $m \in \mathbb{Z}$) with the same initial conditions at $n = 0, 1$. For $n = 1$, this follows from the Wronskian identity (13). Similarly, one shows that

$$y_1(N+m,\lambda) = y_1(N,\lambda)y_1(m,\lambda) + y_1(N+1,\lambda)y_2(m,\lambda), \tag{83}$$

$$y_2(N+m,\lambda) = y_2(N,\lambda)y_1(m,\lambda) + y_2(N+1,\lambda)y_2(m,\lambda) \tag{84}$$

for any $(b,a) \in \mathcal{M}$. Hence, suppressing the variable λ, we get

$$\begin{aligned} y_2(N,S^m b,S^m a) &= \frac{a_m}{a_N}((y_2(N)y_1(m) + y_2(N+1)y_2(m))\, y_1(m) \\ &\quad -(y_1(N)y_1(m) + y_1(N+1)y_2(m))\, y_2(m)) \\ &= \frac{a_m}{a_N}\left(y_2(N)y_1(m)^2 + (y_2(N+1) - y_1(N))y_1(m)y_2(m) - y_1(N+1)y_2(m)^2\right). \end{aligned}$$

By equation (74), this leads to

$$y_2(N,S^m b,S^m a) = -a_m \frac{\partial \Delta_\lambda}{\partial b_m}$$

and formula (77) is established. To prove equation (78), we first conclude from equation (82) that

$$\begin{aligned} \frac{a_N}{a_{m+1}} y_2(N-1,S^{m+1}b,S^{m+1}a) &= y_2(N+m,b,a)y_1(m+1,b,a) \\ &\quad -y_1(N+m,b,a)y_2(m+1,b,a) \end{aligned} \tag{85}$$

and

$$\begin{aligned} \frac{a_N}{a_m} y_2(N+1,S^m b,S^m a) &= y_2(N+m+1,b,a)y_1(m,b,a) \\ &\quad -y_1(N+m+1,b,a)y_2(m,b,a). \end{aligned} \tag{86}$$

Now expand the right-hand sides of equations (85) and (86) according to equations (83) and (84). By equation (75), the sum of equations (85) and (86) is $-a_N \frac{\partial \Delta_\lambda}{\partial a_m}$, thus proving equation (78). ■

As a next step, we compute the gradients of the Dirichlet and periodic eigenvalues. In the following lemma, we consider the fundamental solution $y_1(\cdot, \mu)$ as an N-vector $y_1(j,\mu)_{1\le j\le N}$. Let $\|y_1(\mu)\|^2 = \sum_{j=1}^{N} y_1(j,\mu)^2$, and denote by $\cdot$ the derivative with respect to λ.

Lemma 5.2. If μ is a Dirichlet eigenvalue of $L(b,a)$, then

$$a_N y_1(N,\mu)\dot{y}_1(N+1,\mu) = \|y_1(\mu)\|^2 > 0. \tag{87}$$

In particular, $\dot{y}_1(N+1,\mu) \neq 0$, which implies that all Dirichlet eigenvalues are simple. □

Proof. This follows from adding up the relations (16). ■

As the Dirichlet eigenvalues $(\mu_n)_{1\le n\le N-1}$ of $L(b,a)$ coincide with the roots of $y_1(N+1,\mu)$ and these roots are simple, they are real analytic on $\mathcal{M}$. Similarly, the eigenvalues λ_1 and λ_{2N} are real analytic on $\mathcal{M}$, whereas for any $1 \le n \le N-1$, λ_{2n} and λ_{2n+1} are real analytic on $\mathcal{M} \setminus D_n$. Note that for $(b,a) \in \mathcal{M} \setminus D_n$ and $i \in \{2n, 2n+1\}$, we have $\dot{\Delta}_{\lambda_i} \neq 0$ as λ_i is a simple eigenvalue.

Proposition 5.3. For any $1 \le n \le N-1$, the gradients of the periodic eigenvalues λ_i $(i = 2n, 2n+1)$ on $\mathcal{M} \setminus D_n$ and of the Dirichlet eigenvalues μ_n on $\mathcal{M}$ are given by

$$\nabla_{b,a}\lambda_i = -\frac{\nabla_{b,a}\Delta_\lambda|_{\lambda=\lambda_i}}{\dot{\Delta}_{\lambda_i}} = f_i^2 \quad \text{and} \quad \nabla_{b,a}\mu_n = g_n^2, \tag{88}$$

where we denote by f_i the eigenvector of $L(b,a)$ associated to λ_i, normalized by

$$\sum_{j=1}^{N} f_i(j)^2 = 1 \quad \text{and} \quad (f_i(1), f_i(2)) \in (\mathbb{R}_{>0} \times \mathbb{R}) \cup (\{0\} \times \mathbb{R}_{>0}),$$

and where $g_n = (g_n(j))_{1\le j\le N}$ is the fundamental solution $y_1(\cdot, \mu_n)$ normalized, so that $\sum_{j=1}^{N} g_n(j)^2 = 1$. □

Proof. We first show the second formula in equation (88). Differentiating $y_1(N+1,\mu_n) = 0$ with respect to (b,a), one obtains

$$\nabla_{b,a}\mu_n = -\frac{\nabla_{b,a} y_1(N+1,\lambda)|_{\lambda=\mu_n}}{\dot{y}_1(N+1,\mu_n)}. \tag{89}$$

Here we used that $\dot{y}_1(N+1,\mu_n) \neq 0$ by Lemma 5.2. To compute the gradient $\nabla_{b,a} y_1(N+1,\lambda)|_{\lambda=\mu_n}$, we evaluate equation (69) for $i=1$ and $m=N+1$. In view of $y_1(N+1,\mu_n)=0$ and taking into account equation (26), one then gets

$$\nabla_{b,a}\mu_n = \frac{y_1^2(\mu_n)}{a_N y_1(N,\mu_n)\dot{y}_1(N+1,\mu_n)}. \tag{90}$$

The claimed formula $\nabla_{b,a}\mu_n = g_n^2$ now follows from Lemma 5.2. By differentiating $\Delta_{\lambda_i} = \pm 2$ with respect to (b,a), one obtains $\nabla_{b,a}\lambda_i = -\nabla_{b,a}\Delta_\lambda|_{\lambda=\lambda_i}/\dot{\Delta}_{\lambda_i}$ in a similar fashion. To see that $\nabla_{b,a}\lambda_i = f_i^2$, differentiate $R_{b,a} f_i = \lambda_i f_i$ with respect to (b,a) in the direction $(v,u) \in \mathbb{R}^{2N}$,

$$R_{b,a}\langle \nabla_{b,a} f_i, (v,u)\rangle(k) + (R_{v,u} f_i)(k) = \langle \nabla_{b,a}\lambda_i, (v,u)\rangle f_i(k) + \lambda_i \langle \nabla_{b,a} f_i, (v,u)\rangle(k),$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^{2N}$. Take the scalar product (in $\mathbb{R}^N$) of the above equation with f_i. Now use that

$$\langle \nabla_{b,a} f_i(v,u), R_{b,a} f_i\rangle_{\mathbb{R}^N} = \lambda_i \langle \nabla_{b,a} f_i(v,u), f_i\rangle_{\mathbb{R}^N},$$

$\langle f_i, f_i\rangle_{\mathbb{R}^N} = 1$, and

$$\langle R_{v,u} f_i, f_i\rangle_{\mathbb{R}^N} = \langle f_i^2, (v,u)\rangle_{\mathbb{R}^{2N}},$$

to conclude that $\nabla_{b,a}\lambda_i = f_i^2$ holds. ■

To compute the Poisson brackets involving angle variables, we need to establish some additional auxiliary results. Recall from Section 3 that for $1 \le k,n \le N-1$ with $k \neq n$ and $(b,a) \in \mathcal{M}$, β_k^n is given by

$$\beta_k^n = \int_{\lambda_{2k}}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta_\lambda^2 - 4}}\, d\lambda, \tag{91}$$

whereas

$$\beta_n^n := \eta_n = \int_{\lambda_{2n}}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta_\lambda^2 - 4}}\, d\lambda \pmod{2\pi}. \tag{92}$$

By Theorem 4.2, the functions β_k^n with $k \neq n$ are real analytic on $\mathcal{M}$, whereas β_n^n, when considered mod π, is real analytic on $\mathcal{M} \setminus D_n$.

Proposition 5.4. Let $1 \leq k \leq N-1$ and $(b,a) \in \mathcal{M}$. If $\gamma_k > 0$ and $\lambda_{2k} = \mu_k$, then for any $1 \leq n \leq N-1$,

$$\nabla_{b,a}\beta_k^n = -\frac{\psi_n(\mu_k)}{a_N \dot{\Delta}_{\mu_k}}\, g_k \cdot_{\mathbf{s}} h_k,$$

where h_k denotes the solution of $R_{b,a}\,y = \mu_k\,y$ orthogonal to g_k, that is,

$$\sum_{j=1}^{N} g_k(j) h_k(j) = 0,$$

satisfying the normalization condition $W(h_k, g_k)(N) = 1$. □

Proof. We use a limiting procedure first introduced in [15] for the nonlinear Schrödinger equation and subsequently used for the KdV equation in [11], [12]. We approximate $(b,a) \in \mathcal{M}$ with $\lambda_{2k}(b,a) = \mu_k(b,a) < \lambda_{2k+1}(b,a)$ by $(b',a') \in \mathrm{Iso}(b,a)$, satisfying $\lambda_{2k}(b,a) < \mu_k(b',a') < \lambda_{2k+1}(b,a)$. For such (b',a'), using the substitution $\lambda = \lambda_{2k} + z$ in the integral of equation (91), we obtain

$$\beta_k^n(b',a') = \int_{\lambda_{2k}}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta_\lambda^2 - 4}}\, d\lambda = \int_0^{\mu_k - \lambda_{2k}} \frac{\psi_n(\lambda_{2k} + z)}{\sqrt{z}\sqrt{D(z)}} dz, \tag{93}$$

where $D(z) \equiv D(\lambda_{2k}, z) := (\Delta^2(\lambda_{2k} + z) - 4)/z$. Taking the gradient, undoing the substitution, and recalling definition (29) of the starred square root then leads to

$$\nabla_{b,a}\beta_k^n = \frac{\psi_n(\mu_k)}{\sqrt[*]{\Delta_{\mu_k}^2 - 4}}(\nabla_{b,a}\mu_k - \nabla_{b,a}\lambda_{2k}) + E(b',a'), \tag{94}$$

with the remainder term $E(b', a')$ given by

$$E(b', a') = \int_0^{\mu_k - \lambda_{2k}} \nabla_{b',a'} \left(\frac{\psi_n(\lambda_{2k} + z)}{\sqrt{D(\lambda_{2k}, z)}} \right) \frac{dz}{\sqrt{z}}.$$

As the gradient in the latter integral is a bounded function in z near $z = 0$, locally uniformly in (b', a'), it follows from the dominated convergence theorem that $\lim_{(b',a') \to (b,a)} E(b', a') = 0$.

The gradient $\nabla_{b,a} \beta_k^n$ depends continuously on $(b, a) \in \mathcal{M}$, hence we can conclude by equation (94) that it can be written as

$$\nabla_{b,a} \beta_k^n = \lim_{(b',a') \to (b,a)} \frac{\psi_n(\mu_k)}{\sqrt[*]{\Delta_{\mu_k}^2 - 4}} (\nabla_{b',a'} \mu_k - \nabla_{b',a'} \lambda_{2k}). \tag{95}$$

The gradient of both sides of the Wronskian identity (13)

$$y_1(N, \lambda) y_2(N + 1, \lambda) - y_1(N + 1, \lambda) y_2(N, \lambda) = 1$$

leads to

$$y_1(N + 1) \nabla_{b,a} y_2(N) + y_2(N) \nabla_{b,a} y_1(N + 1) = y_2(N+1) \nabla_{b,a} \Delta + (y_1(N) - y_2(N + 1)) \nabla_{b,a} y_2(N+1). \tag{96}$$

The λ-derivative can then be computed to be

$$y_1(N + 1) \dot{y}_2(N) + \dot{y}_1(N + 1) y_2(N) = y_2(N + 1) \dot{\Delta} + (y_1(N) - y_2(N + 1)) \dot{y}_2(N + 1). \tag{97}$$

Using equations (96), (97), and $y_1(N + 1, \mu_k) = 0$, formula (89) for $\nabla_{b,a} \mu_k$ leads to

$$\nabla_{b,a} \mu_k = - \frac{y_2(N + 1) \nabla_{b,a} \Delta + (y_1(N) - y_2(N + 1)) \nabla_{b,a} y_2(N + 1)}{y_2(N + 1) \dot{\Delta} + (y_1(N) - y_2(N + 1)) \dot{y}_2(N + 1)} \bigg|_{\mu_k}. \tag{98}$$

Further by equation (88),

$$\nabla_{b,a} \lambda_{2k} = - \frac{\nabla_{b,a} \Delta}{\dot{\Delta}} \bigg|_{\lambda = \lambda_{2k}}. \tag{99}$$

Now substitute equations (98) and (99) into $(\nabla_{b',a'}\mu_k - \nabla_{b',a'}\lambda_{2k})$ and use that by equation (29),

$$\sqrt[*]{\Delta^2_{\mu_k} - 4} = (y_1(N) - y_2(N+1))|_{\mu_k}.$$

We claim that

$$\lim_{\substack{(b',a')\\ \to (b,a)}} \frac{\nabla_{b,a}\mu_k - \nabla_{b,a}\lambda_{2k}}{\sqrt[*]{\Delta^2_{\mu_k} - 4}} = \frac{\dot{y}_2(N+1)\nabla_{b,a} y_1(N) - \dot{y}_1(N)\nabla_{b,a} y_2(N+1)}{\dot{\Delta}\, \dot{y}_1(N+1) y_2(N)}\Big|_{\lambda_{2k}}. \tag{100}$$

Indeed, to obtain equation (100) after the above-mentioned substitutions, we split the fraction $\frac{1}{\sqrt[*]{\Delta^2_{\mu_k}-4}}(\nabla_{b',a'}\mu_k - \nabla_{b',a'}\lambda_{2k})$ into two parts which are treated separately. In the first part, we collect all terms in $\frac{1}{\sqrt[*]{\Delta^2_{\mu_k}-4}}(\nabla_{b',a'}\mu_k - \nabla_{b',a'}\lambda_{2k})$ which contain $(y_1(N) - y_2(N+1))|_{\mu_k}$ in the nominator, and get after cancellation,

$$I(a', b') := \frac{-\dot{\Delta}|_{\lambda_{2k}} \cdot \nabla_{b',a'} y_2(N+1)|_{\mu_k} + \nabla_{b',a'}\Delta|_{\lambda_{2k}} \cdot \dot{y}_2(N+1)|_{\mu_k}}{\dot{\Delta}|_{\lambda_{2k}} \cdot (y_2(N+1)\dot{\Delta} + (y_1(N) - y_2(N+1))\dot{y}_2(N+1))|_{\mu_k}}.$$

Using again equation (97), we then get

$$\lim_{(b',a')\to(b,a)} I(b', a') = \frac{\dot{y}_2(N+1)\nabla_{b,a} y_1(N) - \dot{y}_1(N)\nabla_{b,a} y_2(N+1)}{\dot{\Delta}\dot{y}_1(N+1) y_2(N)}\Big|_{\lambda_{2k}}.$$

The second term is then given by

$$II(b', a') = \frac{y_2(N+1)|_{\mu_k} \cdot (\nabla_{b',a'}\Delta|_{\lambda_{2k}} \cdot \dot{\Delta}|_{\mu_k} - \dot{\Delta}|_{\lambda_{2k}} \cdot \nabla_{b',a'}\Delta|_{\mu_k})}{\dot{\Delta}|_{\lambda_{2k}} \cdot (y_1(N) - y_2(N+1))|_{\mu_k} \cdot (\dot{y}_1(N+1) y_2(N))|_{\mu_k}}.$$

Note that the nominator of $II(b', a')$ is of the order $O(\mu_k - \lambda_{2k})$. In view of equation (29), we have

$$(y_1(N) - y_2(N+1))|_{\mu_k} = O(\sqrt{\mu_k - \lambda_{2k}}),$$

whereas the other terms in the denominator of $II(b', a')$ are bounded away from zero. Indeed, λ_{2k} being a simple eigenvalue for (b, a) means $\dot{\Delta}|_{\lambda_{2k}} \neq 0$ for (b', a') near (b, a). Further, use a version of equation (97) in the case $\lambda_{2k} = \mu_k$ to conclude that

$$\dot{y}_1(N+1) y_2(N) = y_2(N+1)\dot{\Delta}_{\lambda_{2k}}.$$

Hence $\dot{y}_1(N+1)y_2(N)|_{\mu_k} \neq 0$ for (b', a') near (b, a) and $II(b', a')$ vanishes in the limit of $\mu_k \to \lambda_{2k}$.

Substituting equations (80) and (81) into equation (100), we obtain

$$\left.\frac{\dot{y}_2(N+1)\nabla_{b,a}\, y_1(N) - \dot{y}_1(N)\nabla_{b,a}\, y_2(N+1)}{\dot{\Delta}\, \dot{y}_1(N+1)y_2(N)}\right|_{\mu_k} = -\frac{1}{a_N \dot{\Delta}}\, y_1 \cdot \mathbf{s}\, y_0$$

with $y_0 = \frac{\dot{y}_2(N+1)}{\dot{y}_1(N+1)} y_1 - y_2$. Hence

$$\nabla_{b,a}\beta_k^n = -\frac{\psi_n(\mu_k)}{a_N \dot{\Delta}(\mu_k)}\, y_1 \cdot \mathbf{s}\, y_0.$$

Since β_k^n is invariant under the translation $b \mapsto b + t(1, \ldots, 1)$, the scalar product $\langle \nabla_{b,a}\beta_k^n, (\mathbf{1}, \mathbf{0})\rangle_{\mathbb{R}^{2N}}$ vanishes. Hence

$$0 = \sum_{j=1}^{N} \frac{\partial \beta_k^n}{\partial b_j} = -\frac{\psi_n(\mu_k)}{a_N \dot{\Delta}(\mu_k)} \sum_{j=1}^{N} y_1(j) y_0(j).$$

It means that y_1 and y_0 are orthogonal to each other. Finally, we introduce $h_k := \|y_1\| y_0$ and verify that

$$W(h_k, g_k) = W\left(\|y_1\| y_0, \frac{y_1}{\|y_1\|}\right) = W(y_0, y_1) = W\left(\frac{\dot{y}_2(N+1)}{\dot{y}_1(N+1)} y_1 - y_2, y_1\right).$$

By equation (13), it then follows that

$$W(h_k, g_k) = -W(y_2, y_1) = W(y_1, y_2).$$

Hence by equation (14),

$$W(h_k, g_k)(N) = W(y_1, y_2)(N) = 1.$$

This completes the proof of Proposition 5.4. ■

6 Orthogonality Relations

In Propositions 5.1, 5.3, and 5.4, we have expressed the gradients of Δ_λ, μ_n, and, on a subset of $\mathcal{M}$, of β_k^n in terms of products of fundamental solutions of the difference equation

(11). In this section, we establish orthogonality relations between such products—see [2] for similar computations. Recall that in equation (72), we have introduced for arbitrary sequences $(v_j)_{j\in\mathbb{Z}}$, $(w_j)_{j\in\mathbb{Z}}$ the $2N$-vector $v \cdot_{\mathbf{S}} w$.

Lemma 6.1. For any $(b, a) \in \mathcal{M}$, let v_1, w_1 and v_2, w_2 be pairs of solutions of equation (11) for arbitrarily given real numbers μ and λ, respectively. Then

$$\frac{2(\lambda-\mu)}{a_1 a_N}\langle v_1 \cdot_{\mathbf{S}} w_1, J(v_2 \cdot_{\mathbf{S}} w_2)\rangle = V + B, \tag{101}$$

where

$$V := (W_1 \cdot S\,W_2)\Big|_0^N + (S\,W_1 \cdot W_2)\Big|_0^N \tag{102}$$

with W_1 and W_2 denoting the Wronskians $W_1 := W(v_1, w_2)$, $W_2 := W(w_1, v_2)$, and where B is given by

$$B := \frac{(\lambda-\mu)}{a_1}\Big((v_1 \cdot w_1)|_1^{N+1}(v_2 \cdot_{\mathbf{S}} w_2)(2N) - (v_2 \cdot w_2)|_1^{N+1}(v_1 \cdot_{\mathbf{S}} w_1)(2N)\Big). \tag{103}$$

□

Proof. We prove equation (101) by a straightforward calculation, using the recurrence property (15) of the Wronskian sequences W_1 and W_2. By definition (3) of J, we can write

$$2\,\langle v_1 \cdot_{\mathbf{S}} w_1, J(v_2 \cdot_{\mathbf{S}} w_2)\rangle = E_1 + B_1,$$

where

$$E_1 := \sum_{k=1}^{N} a_k [(v_1 \cdot_{\mathbf{S}} w_1)(k)(v_2 \cdot_{\mathbf{S}} w_2)(N+k) - (v_1 \cdot_{\mathbf{S}} w_1)(k+1)(v_2 \cdot_{\mathbf{S}} w_2)(N+k)$$
$$- (v_1 \cdot_{\mathbf{S}} w_1)(N+k)(v_2 \cdot_{\mathbf{S}} w_2)(k) + (v_1 \cdot_{\mathbf{S}} w_1)(N+k)(v_2 \cdot_{\mathbf{S}} w_2)(k+1)]$$

and

$$B_1 := a_N\,((v_1 \cdot_{\mathbf{S}} w_1)(N+1) - (v_1 \cdot_{\mathbf{S}} w_1)(1))\,(v_2 \cdot_{\mathbf{S}} w_2)(2N)$$
$$+ a_N\,((v_2 \cdot_{\mathbf{S}} w_2)(1) - (v_2 \cdot_{\mathbf{S}} w_2)(N+1))\,(v_1 \cdot_{\mathbf{S}} w_1)(2N).$$

Let us first consider E_1. Calculating the products $v_j \cdot_{\mathbf{s}} w_j$ according to equation (72), we obtain, after regrouping,

$$E_1 = \sum_{k=1}^{N} a_k \big[(v_2(k) w_1(k) + v_2(k+1) w_1(k+1)) W_1(k) \\ + (v_1(k) w_2(k) + v_1(k+1) w_2(k+1)) W_2(k) \big] + B_2$$

with

$$B_2 := a_N (v_1(N+1) w_1(N+1) - (v_1 \cdot_{\mathbf{s}} w_1)(N+1))\,(v_2 \cdot_{\mathbf{s}} w_2)(2N) \\ + a_N ((v_2 \cdot_{\mathbf{s}} w_2)(N+1) - v_2(N+1) w_2(N+1))\,(v_1 \cdot_{\mathbf{s}} w_1)(2N).$$

Multiply E_1 by $(\lambda - \mu)$ and use the recurrence relation (15) to express $(\lambda - \mu) v_2(k) w_1(k)$, $(\lambda - \mu) v_2(k+1) w_1(k+1)$, $(\lambda - \mu) v_1(k) w_2(k)$, and $(\lambda - \mu) v_1(k+1) w_2(k+1)$ in terms of the Wronskians W_1 and W_2 to get

$$(\lambda - \mu) E_1 = \sum_{k=1}^{N} [a_k a_{k+1} (W_1(k) W_2(k+1) + W_1(k+1) W_2(k)) - a_{k-1} a_k (W_1(k-1) W_2(k) \\ + W_1(k) W_2(k-1))] + (\lambda - \mu) B_2.$$

The sum on the right-hand side of the latter identity is a telescoping sum and equals the term $a_1 a_N V$ with V defined in equation (102). In a straightforward way, one sees that $\frac{(\lambda-\mu)}{a_1 a_N}(B_1 + B_2)$ equals the expression B defined by equation (103), hence formula (101) is established. ■

Corollary 6.2. For any $\lambda, \mu \in \mathbb{C}$,

$$\{\Delta_\lambda, \Delta_\mu\}_J = 0. \tag{104}$$

□

Proof. By formula (76) for the gradient of Δ_λ,

$$\{\Delta_\lambda, \Delta_\mu\}_J = \langle \nabla_{b,a} \Delta_\lambda, J \nabla_{b,a} \Delta_\mu \rangle$$

is a linear combination of terms of the form $\langle v_1 \cdot_{\mathbf{s}} w_1, J(v_2 \cdot_{\mathbf{s}} w_2) \rangle$ for pairs of fundamental solutions v_1, w_1 and v_2, w_2 of equation (11) for μ and λ, respectively. In view of equations (77) and (78), $\nabla_b \Delta_\lambda$ and $\nabla_a \Delta_\lambda$ are both N-periodic. In the case $\lambda \neq \mu$, we use Lemma 6.1 and note that the boundary terms (102) and (103) in Lemma 6.1 vanish, hence $\{\Delta_\lambda, \Delta_\mu\}_J = 0$. In the case $\lambda = \mu$, identity (104) follows from the skew symmetry of $\{\cdot, \cdot\}_J$. ■

Corollary 6.3. Let $1 \le k \le 2N$ and $\lambda \in \mathbb{C}$. On the open subset of $\mathcal{M}$ where λ_k is a simple eigenvalue of $Q(b, a)$, one has

$$\{\lambda_k, \Delta_\lambda\}_J = 0.$$

□

Proof. Using formula (88) for $\nabla_{b,a}\lambda_k$, we conclude from Corollary 6.2 that

$$\{\lambda_k, \Delta_\lambda\}_J = -\frac{1}{\dot{\Delta}_{\lambda_k}}\{\Delta_\mu, \Delta_\lambda\}_J|_{\mu=\lambda_k} = 0.$$

■

Corollary 6.4. Let μ_n be the nth Dirichlet eigenvalue of $L(b, a)$ and $\lambda \neq \mu_n$ a real number. Then

$$(\lambda - \mu_n)\langle Y_1^2(\mu_n), JY_1^2(\lambda)\rangle = \left(a_N \frac{y_1(N+1,\lambda)}{y_2(N+1,\mu_n)}\right)^2, \tag{105}$$

$$(\lambda - \mu_n)\langle Y_1^2(\mu_n), JY_1(\lambda)\cdot_{\mathbf{s}} Y_2(\lambda)\rangle = a_N^2 \frac{y_1(N+1,\lambda)y_2(N+1,\lambda)}{y_2(N+1,\mu_n)^2}, \tag{106}$$

$$(\lambda - \mu_n)\langle Y_1^2(\mu_n), JY_2^2(\lambda)\rangle = a_N^2\left(\left(\frac{y_2(N+1,\lambda)}{y_2(N+1,\mu_n)}\right)^2 - 1\right). \tag{107}$$

□

Proof. The three stated identities follow from Lemma 6.1, using that $y_1(N+1, \mu_n) = 0$, $y_1(2, \mu_n) = -a_N/a_1$, and, by the Wronskian identity (26), $y_1(N, \mu_n) \cdot y_2(N+1, \mu_n) = 1$. ■

Corollary 6.5. Let μ_n be the nth Dirichlet eigenvalue of $L(b, a)$ and $\lambda \neq \mu_n$ a real number. Then

$$\{\mu_n, \Delta_\lambda\}_J = \frac{y_1(N+1,\lambda)}{\dot{y}_1(N+1,\mu_n)} \frac{\sqrt[*]{\Delta_{\mu_n}^2 - 4}}{\lambda - \mu_n}. \tag{108}$$

□

Proof. By equation (90), combined with equation (26), we get

$$\{\mu_n, \Delta_\lambda\}_J = \frac{y_2(N+1,\mu_n)}{a_N \dot{y}_1(N+1,\mu_n)} \big\langle y_1^2(\mu_n), J\nabla_{b,a}\Delta_\lambda \big\rangle. \tag{109}$$

Substituting formula (76) for $J\nabla_{b,a}\Delta_\lambda$, we obtain

$$\begin{aligned}\big\langle y_1^2(\mu_n), J\nabla_{b,a}\Delta_\lambda\big\rangle = &-\frac{1}{a_N} y_2(N,\lambda)\big\langle y_1^2(\mu_n), J y_1^2(\lambda)\big\rangle \\ &-\frac{1}{a_N}(y_2(N+1,\lambda) - y_1(N,\lambda))\big\langle y_1^2(\mu_n), J y_1(\lambda) \cdot_{\mathbf{s}} y_2(\lambda)\big\rangle \\ &+\frac{1}{a_N} y_1(N+1,\lambda)\big\langle y_1^2(\mu_n), J y_2^2(\lambda)\big\rangle. \end{aligned} \tag{110}$$

To evaluate the right side of equation (110), we apply Corollary 6.4 and get

$$\begin{aligned}\frac{-\mu_n}{a_1 a_N}\big\langle y_1^2(\mu_n), J\nabla_{b,a}\Delta_\lambda\big\rangle = &\frac{1}{a_1 y_2(N+1,\mu_n)^2}\big(- y_2(N,\lambda) y_1(N+1,\lambda)^2 \\ &-(y_2(N+1,\lambda) - y_1(N,\lambda)) y_1(N+1,\lambda) y_2(N+1,\lambda) \\ &+ y_1(N+1,\lambda) y_2(N+1,\lambda)^2\big) - \frac{y_1(N+1,\lambda)}{a_1}. \end{aligned}$$

Using the Wronskian identity (14), the sum of the terms in the square bracket of the latter expression simplifies, and one obtains

$$\begin{aligned}\frac{\lambda-\mu_n}{a_1 a_N}\langle y_1^2(\mu_n), J\nabla_{b,a}\Delta_\lambda\rangle &= \frac{y_1(N+1,\lambda)}{a_1 y_2(N+1,\mu_n)^2} - \frac{y_1(N+1,\lambda)}{a_1} \\ &= \frac{y_1(N+1,\lambda)}{a_1}(y_1(N,\mu_n)^2 - 1), \end{aligned} \tag{111}$$

where for the latter equality, we again used equation (14). Substituting equation (111) into equation (109), we get

$$\begin{aligned}\frac{\lambda-\mu_n}{a_1 a_N}\{\mu_n, \Delta_\lambda\}_J &= \frac{y_2(N+1,\mu_n) y_1(N+1,\lambda)}{a_1 a_N \dot{y}_1(N+1,\mu_n)}(y_1(N,\mu_n)^2 - 1) \\ &= \frac{y_1(N+1,\lambda)}{a_1 a_N \dot{y}_1(N+1,\mu_n)}\sqrt[*]{\Delta_{\mu_n}^2 - 4}, \end{aligned}$$

where we used that, by the definition of the starred square root (29),

$$\sqrt[*]{\Delta_{\mu_n}^2 - 4} = y_1(N, \mu_n) - y_2(N+1, \mu_n) = y_2(N+1, \mu_n)(y_1(N, \mu_n)^2 - 1).$$

This proves equation (108). ■

Proposition 6.6. For any $\lambda \in \mathbb{R}$, $1 \le n \le N-1$, and $(b, a) \in \mathcal{M} \setminus D_n$,

$$\{\theta_n, \Delta_\lambda\}_J = \psi_n(\lambda). \tag{112}$$

□

Proof. Recall that $\theta_n = \sum_{k=1}^{N-1} \beta_k^n \pmod{2\pi}$ with β_k^n given by equations (91)–(92). To compute $\{\beta_k^n, \Delta_\lambda\}_J$, we first consider the case where $(b, a) \notin \bigcup_{k=1}^{N-1} D_k$ and $\lambda_{2k} < \mu_k < \lambda_{2k+1}$ for any $1 \le k \le N-1$. Then λ_{2k} and μ_k^* are smooth near (b, a) and, by Leibniz's rule, we get

$$\{\beta_k^n, \Delta_\lambda\}_J = \left(\int_{\lambda_{2k}}^{\mu_k^*} \left\{ \frac{\psi_n(\mu)}{\sqrt{\Delta_\mu^2 - 4}}, \Delta_\lambda \right\}_J d\mu \right.$$
$$\left. + \frac{\psi_n(\mu_k)}{\sqrt[*]{\Delta_{\mu_k}^2 - 4}} \{\mu_k, \Delta_\lambda\}_J - \frac{\psi_n(\lambda_{2k})}{\sqrt[*]{\Delta_{\lambda_{2k}}^2 - 4}} \{\lambda_{2k}, \Delta_\lambda\}_J \right).$$

By Corollary 6.3, $\{\lambda_{2k}, \Delta_\lambda\}_J = 0$. Moreover, as the gradient $\nabla_{b,a} \frac{\psi_n(\mu)}{\sqrt{\Delta_\mu^2 - 4}}$ is orthogonal to $T_{b,a}\,\mathrm{Iso}\,(b, a)$ and $J\nabla_{b,a}\Delta_\lambda \in T_{b,a}\,\mathrm{Iso}\,(b, a)$, it follows that the Poisson bracket $\{\frac{\psi_n(\mu)}{\sqrt{\Delta_\mu^2 - 4}}, \Delta_\lambda\}_J$ vanishes for any μ in the isolating neighborhood U_n of G_n. Hence

$$\{\beta_k^n, \Delta_\lambda\}_J = \frac{\psi_n(\mu_k)}{\sqrt[*]{\Delta_{\mu_k}^2 - 4}} \{\mu_k, \Delta_\lambda\}_J.$$

By equation (108), we then obtain

$$\{\theta_n, \Delta_\lambda\}_J = \sum_{k=1}^{N-1} \frac{\psi_n(\mu_k)}{\dot{y}_1(N+1, \mu_k)} \frac{y_1(N+1, \lambda)}{\lambda - \mu_k} = \psi_n(\lambda),$$

where for the latter equality, we used that $\sum_{k=1}^{N-1} \frac{\psi_n(\mu_k)}{\dot{y}_1(N+1,\mu_k)} \frac{y_1(N+1,\lambda)}{\lambda-\mu_k}$ and $\psi_n(\lambda)$ are both polynomials in λ of degree at most $N-2$ which agree at the $N-1$ points $(\mu_k)_{1 \le k \le N-1}$.

In the general case, where $(b, a) \in \mathcal{M} \setminus D_n$ and the Dirichlet eigenvalues are arbitrary, $\lambda_{2k} \le \mu_k \le \lambda_{2k+1}$ for any $1 \le k \le N-1$, the claimed result follows from the case treated above by continuity. ■

Proposition 6.7. Let $1 \le n, m, k, l \le N-1$ and let $(b,a) \in \mathcal{M}$ with $\lambda_{2i}(b,a) = \mu_i(b,a)$ for $i = k, l$. Then

$$\{\beta_k^n, \beta_l^m\}_J = 0.$$

□

Proof. In view of Proposition 5.4, this amounts to showing that the scalar product $\langle (g_k \cdot_{\mathbf{S}} h_k), J(g_l \cdot_{\mathbf{S}} h_l)\rangle$ vanishes. For $k = l$, this follows from the skew symmetry of the Poisson bracket, hence we can assume that $k \neq l$. We apply Lemma 6.1 with $v_1 := g_k$, $w_1 := h_k$, $v_2 := h_l$, and $w_2 := g_l$, which implies that $W_1 = W(g_k, g_l)$ and $W_2 = W(h_k, h_l)$. Since $g_k(1)$, $g_l(1)$, $g_k(N+1)$, and $g_l(N+1)$ all vanish, we conclude that $W_1(N) = W_1(0) = 0$ and $(SW_1)(N) = (SW_1)(0) = 0$, hence the expressions V and E, defined in equations (102) and (103) vanish. This proves the claim. ■

7 Canonical Relations

In this section, we complete the proof of Theorem 1.1 and Corollary 1.2. In particular, we show that the variables $(I_n)_{1 \le n \le N-1}$, $(\theta_n)_{1 \le n \le N-1}$ satisfy the canonical relations stated in Theorem 1.1.

Using the results of the preceding sections, we can now compute the Poisson brackets among the action and angle variables introduced in Section 3.

Theorem 7.1. The action–angle variables $(I_n)_{1 \le n \le N-1}$ and $(\theta_n)_{1 \le n \le N-1}$ satisfy the following canonical relations for $1 \le n, m \le N-1$:

(i) on $\mathcal{M}$,

$$\{I_n, I_m\}_J = 0; \tag{113}$$

(ii) on $\mathcal{M} \setminus D_n$,

$$\{\theta_n, I_m\}_J = -\{I_m, \theta_n\}_J = -\delta_{nm}. \tag{114}$$

□

Proof. Recall that $\frac{d}{dt} \operatorname{arcosh}(t) = (t^2 - 1)^{-\frac{1}{2}}$. Hence for any $(b,a) \in \mathcal{M}$,

$$I_n = \frac{1}{2\pi} \int_{\Gamma_n} \lambda \frac{d}{d\lambda} \operatorname{arcosh} \left| \frac{\Delta_\lambda}{2} \right| d\lambda$$

and therefore

$$\nabla_{b,a} I_n = \frac{1}{2\pi} \int_{\Gamma_n} \lambda \frac{d}{d\lambda} \frac{\nabla_{b,a} \Delta_\lambda}{\sqrt[c]{\Delta_\lambda^2 - 4}} d\lambda.$$

Integrating by parts we get

$$\nabla_{b,a} I_n = -\frac{1}{2\pi} \int_{\Gamma_n} \frac{\nabla_{b,a} \Delta_\lambda}{\sqrt[c]{\Delta_\lambda^2 - 4}} d\lambda. \tag{115}$$

As $\{\Delta_\lambda, \Delta_\mu\}_J = 0$ for all $\lambda, \mu \in \mathbb{C}$ by Corollary 6.2, it follows that $\{I_n, I_m\}_J = 0$ on $\mathcal{M}$ for any $1 \leq n, m \leq N-1$.

To prove equation (114), use equation (115) and then Proposition 6.6 to get

$$\{\theta_n, I_m\}_J = -\frac{1}{2\pi} \int_{\Gamma_m} \frac{\{\theta_n, \Delta_\lambda\}_J}{\sqrt[c]{\Delta_\lambda^2 - 4}} d\lambda = -\frac{1}{2\pi} \int_{\Gamma_m} \frac{\psi_n(\lambda)}{\sqrt[c]{\Delta_\lambda^2 - 4}} d\lambda = -\delta_{nm}$$

by the normalizing condition (36) of ψ_n. ∎

To prove that the angles $(\theta_n)_{1 \leq n \leq N-1}$ pairwise Poisson commute, we need the following lemma. We denote by $K = K(b,a)$ the index set of the open gaps, that is,

$$K(b,a) = \{1 \leq n \leq N-1 : \gamma_n(b,a) > 0\}.$$

Lemma 7.2. At every point (b,a) in $\mathcal{M}$, the set of vectors

(i) $$((\nabla_{b,a} I_n)_{n \in K}, \nabla_{b,a} C_1, \nabla_{b,a} C_2)$$

and

(ii) $$(J\nabla_{b,a} I_n)_{n \in K}$$

are both linearly independent. □

Proof. The claimed statements follow from the orthogonality relations stated in Theorem 7.1: Let $(b,a) \in \mathcal{M}$ and suppose that for some real coefficients $(r_n)_{n \in K} \subseteq \mathbb{R}$ and $s_1, s_2 \in \mathbb{R}$ we have

$$\sum_{n \in K} r_n \nabla_{b,a} I_n + s_1 \nabla_{b,a} C_1 + s_2 \nabla C_2 = 0.$$

For any $m \in K$, take the scalar product of this identity with $J\nabla_{b,a}\theta_m$. Using that $\{I_n, \theta_m\}_J = \delta_{nm}$ and that C_1 and C_2 are Casimir functions of $\{\cdot,\cdot\}_J$, one obtains

$$0 = \sum_{n\in K} r_n\{I_n, \theta_m\}_J = \sum_{n\in K} r_n\delta_{nm} = r_m.$$

Thus $r_m = 0$ for all $m \in K$, and it follows that $s_1\nabla_{b,a}C_1 + s_2\nabla_{b,a}C_2 = 0$. By equations (8) and (9), $\nabla_{b,a}C_1$ and $\nabla_{b,a}C_2$ are linearly independent, hence $s_1 = s_2 = 0$. This proves (i). The proof of (i) also shows that (ii) holds. ■

Theorem 7.3. In addition to the canonical relations stated in Theorem 7.1, the angle variables $(\theta_n)_{1\le n\le N-1}$ satisfy for any $1 \le n, m \le N-1$ on $\mathcal{M} \setminus (D_n \cup D_m)$

$$\{\theta_n, \theta_m\}_J = 0. \tag{116}$$

□

Proof. Let $1 \le n, m \le N-1$. By continuity, it suffices to prove identity (116) for $(b,a) \in \mathcal{M} \setminus \left(\bigcup_{l=1}^{N-1} D_l\right)$. Let (b,a) be an arbitrary element in $\mathcal{M} \setminus \left(\bigcup_{l=1}^{N-1} D_l\right)$. Recall that $\mathrm{Iso}(b,a)$ denotes the set of all elements (b',a') in $\mathcal{M}$ with $\mathrm{spec}(Q_{b',a'}) = \mathrm{spec}(Q_{b,a})$,

$$\mathrm{Iso}\,(b,a) = \{(b',a') \in \mathcal{M} : \Delta(\cdot, b', a') = \Delta(\cdot, b, a)\}.$$

Then $\mathrm{Iso}(b,a)$ is a torus contained in $\mathcal{M} \setminus (\bigcup_{l=1}^{N-1} D_l)$, and as all eigenvalues of $Q(b,a)$ are simple, its dimension is $N-1$. By Lemma 7.2, at any point $(b',a') \in \mathrm{Iso}(b,a)$, the vectors $(J\,\nabla_{b',a'} I_k)_{1\le k\le N-1}$ are linearly independent. Using formula (115) for the gradient of I_k, one sees that, by Corollary 6.2, for any $\mu \in \mathbb{R}$, $1 \le k \le N-1$,

$$\langle \nabla_{b',a'}\Delta_\mu, J\,\nabla_{b',a'} I_k\rangle = -\frac{1}{2\pi}\int_{\Gamma_n} \frac{\{\Delta_\mu, \Delta_\lambda\}_J}{\sqrt[c]{\Delta_\lambda^2 - 4}}\, d\lambda = 0.$$

Hence for any $(b',a') \in \mathrm{Iso}(b,a)$,

$$(J\,\nabla_{b',a'} I_k)_{1\le k\le N-1} \in T_{b',a'}\,\mathrm{Iso}\,(b,a),$$

and therefore these vectors form a basis of $T_{b',a'}\,\mathrm{Iso}\,(b,a)$.

To prove identity (116), we apply the Jacobi identity

$$\{F,\{G,H\}_J\}_J+\{G,\{H,F\}_J\}_J+\{H,\{F,G\}_J\}_J=0$$

to the functions I_k, θ_n, and θ_m. Since by Theorem 7.1 $\{I_k,\theta_n\}_J=\delta_{kn}$, we obtain

$$\{I_k,\{\theta_n,\theta_m\}_J\}_J=0 \text{ on } \mathcal{M}\setminus\left(\bigcup_{l=1}^{N-1}D_l\right) \text{ for any } 1\le k\le N-1.$$

It then follows by the above considerations that $\nabla_{b',a'}\{\theta_n,\theta_m\}_J$ is orthogonal to $T_{b',a'}\,\mathrm{Iso}\,(b,a)$ for all $(b',a')\in\mathrm{Iso}(b,a)$, i.e. $\{\theta_n,\theta_m\}_J$ is constant on $\mathrm{Iso}(b,a)$,

$$\{\theta_n,\theta_m\}_J(b',a')=\{\theta_n,\theta_m\}_J(b,a)\quad \forall\,(b',a')\in\ \mathrm{Iso}\,(b,a).$$

By [16], Theorem 2.1, there exists a unique element $(b',a')\in\mathrm{Iso}(b,a)$ satisfying $\mu_k(b',a')=\lambda_{2k}(b,a)$ for all $1\le k\le N-1$. The claimed identity (116) then follows from Proposition 6.7. ■

Proof of Theorem 1.1 By Theorems 3.5 and 4.2, the action and angle variables introduced in Definitions 3.1 and 4.1, respectively, have the claimed analyticity properties. The canonical relations among these variables have been verified in Theorems 7.1 and 7.3, and the relations $\{C_i,I_n\}_J=0$ (on $\mathcal{M}$) and $\{C_i,\theta_n\}_J=0$ (on $\mathcal{M}\setminus D_n$) follow from the fact that C_1 and C_2 are Casimir functions. It remains to show that the actions Poisson commute with the Toda Hamiltonian. To this end, note that that the Hamiltonian H can be written as

$$H=\frac{1}{2}\sum_{n=1}^{N}b_n^2+\sum_{n=1}^{N}a_n^2=\frac{1}{2}\,\mathrm{tr}\,(L(b,a)^2)=\frac{1}{2}\sum_{j=1}^{N}(\lambda_j^+)^2,$$

where $(\lambda_j^+)_{1\le j\le N}$ are the N eigenvalues of $L(b,a)$. Recall that on the dense open subset $\mathcal{M}\setminus\cup_{k=1}^{N-1}D_k$ of $\mathcal{M}$, the λ_i^+'s $(1\le i\le N)$ are simple eigenvalues and hence real analytic. It then follows by equation (115) that for any $1\le n\le N-1$,

$$\{H,I_n\}_J=\sum_{i=1}^{N}\lambda_i^+\{\lambda_i^+,I_n\}_J=-\sum_{i=1}^{N}\frac{\lambda_i^+}{2\pi}\int_{\Gamma_n}\frac{\{\lambda_i^+,\Delta_\lambda\}_J}{\sqrt[c]{\Delta_\lambda^2-4}}d\lambda=0,$$

where for the latter identity we used Corollary 6.3. Hence for any $1 \le n \le N-1$,

$$\{H, I_n\}_J = 0 \quad \text{on } \mathcal{M} \setminus \cup_{k=1}^{N-1} D_k.$$

By continuity, it then follows that $\{H, I_n\}_J = 0$ everywhere on $\mathcal{M}$. ■

Proof of Corollary 1.2 Since for any $\beta \in \mathbb{R}$ and $\alpha > 0$, the symplectic leaf $\mathcal{M}_{\beta,\alpha}$ is a submanifold of $\mathcal{M}$ of dimension $2(N-1)$, there are at most $N-1$ independent integrals in involution on $\mathcal{M}_{\beta,\alpha}$. For any given $(b, a) \in \mathcal{M}_{\beta,\alpha}$, let $\pi_{\beta,\alpha}$ denote the orthogonal projection $T_{b,a}\mathcal{M} \to T_{b,a}\mathcal{M}_{\beta,\alpha}$. Then the gradient of the restriction $I_n|_{\mathcal{M}_{\beta,\alpha}}$ of I_n to $\mathcal{M}_{\beta,\alpha}$ $(1 \le n \le N-1)$ is given by $\pi_{\beta,\alpha}\nabla_{b,a} I_n$. By Lemma 7.2, the vectors $(\pi_{\beta,\alpha}\nabla_{b,a} I_n)_{n\in K}$ are linearly independent. As $\mathcal{M}_{\beta,\alpha} \setminus \cup_{k=1}^{N-1} D_k$ is dense in $\mathcal{M}_{\beta,\alpha}$, it then follows that $(I_n|_{\mathcal{M}_{\beta,\alpha}})_{1\le n\le N-1}$ are functionally independent. Finally, as C_1 and C_2 are Casimir functions of $\{\cdot,\cdot\}_J$, it follows that for any $(b, a) \in \mathcal{M}_{\beta,\alpha}$,

$$\begin{aligned}\{I_n|_{\mathcal{M}_{\beta,\alpha}}, I_m|_{\mathcal{M}_{\beta,\alpha}}\}_J(b,a) &= \langle \pi_{\beta,\alpha}\nabla_{b,a} I_n, \pi_{\beta,\alpha} J\nabla_{b,a} I_m\rangle \\ &= \langle \nabla_{b,a} I_n, J\nabla_{b,a} I_m\rangle = \{I_n, I_m\}_J = 0,\end{aligned}$$

that is, the restrictions $I_n|_{\mathcal{M}_{\beta,\alpha}}$ of I_n $(1 \le n \le N-1)$ are in involution. ■

A Appendix Proof of Lemma 3.7

In this Appendix, we prove Lemma 3.7. It turns out that the proof in ([1], pp. 601–602) of the special case where the parameter α in equation (1) equals 1 can be adapted for arbitrary values.

Proof of Lemma 3.7 Let (b, a) be an arbitrary element in $\mathcal{M}$ and $1 \le n \le N-1$. First note that $I_n = \frac{1}{\pi}\int_{\lambda_{2n}}^{\lambda_{2n+1}} \operatorname{arcosh}|\frac{1}{2}\Delta(\lambda)|\, d\lambda$ and use $\frac{d}{dt}\operatorname{arcosh}(t) = \frac{1}{\sqrt{t^2-1}}$ to obtain

$$I_n = \frac{1}{\pi}\int_{\lambda_{2n}}^{\lambda_{2n+1}} \int_1^{|\Delta(\lambda)|/2} \frac{1}{\sqrt{t^2-1}}\, dt\, d\lambda.$$

Since the integrand of the inner integral is nonincreasing, we estimate it by its value at $\frac{|\Delta(\lambda)|}{2}$. This leads to

$$I_n \ge \frac{1}{\pi}\int_{\lambda_{2n}}^{\lambda_{2n+1}} \frac{\sqrt{|\Delta(\lambda)|-2}}{\sqrt{|\Delta(\lambda)|+2}}\, d\lambda. \tag{A.1}$$

We will show that for $\lambda_{2n} \le \lambda \le \lambda_{2n+1}$,

$$\frac{\sqrt{|\Delta(\lambda)| - 2}}{\sqrt{|\Delta(\lambda)| + 2}} \ge \frac{\sqrt{\lambda - \lambda_{2n}}\sqrt{\lambda_{2n+1} - \lambda}}{\lambda_{2N} - \lambda_1}. \tag{A.2}$$

We then substitute equation (A.2) into the integral (A.1) and split the integration interval into two equal parts,

$$I_n \ge \frac{2}{\pi} \frac{1}{\lambda_{2N} - \lambda_1} \int_{\lambda_{2n}}^{\tau_n} \frac{\sqrt{\lambda - \lambda_{2n}}\sqrt{\lambda_{2n+1} - \lambda}}{\lambda_{2N} - \lambda_1}\, d\lambda,$$

where $\tau_n = (\lambda_{2n} + \lambda_{2n+1})/2$. For $\lambda_{2n} \le \lambda \le \tau_n$, we estimate the quantity $\lambda_{2n+1} - \lambda$ from below by $\gamma_n/2$, yielding

$$I_n \ge \frac{2}{\pi} \frac{1}{\lambda_{2N} - \lambda_1} \int_{\lambda_{2n}}^{\tau_n} \sqrt{\frac{\gamma_n}{2}} \sqrt{\lambda - \lambda_{2n}}\, d\lambda = \frac{1}{3\pi(\lambda_{2N} - \lambda_1)}\, \gamma_n^2.$$

It remains to verify equation (A.2). If λ_{2n} and λ_{2n+1} are periodic eigenvalues, we have $\Delta(\lambda) \ge 2$ for $\lambda_{2n} \le \lambda \le \lambda_{2n+1}$, i.e. $|\Delta(\lambda)| = \Delta(\lambda)$. In order to make writing easier, let us assume that N is even—the case where N is odd is treated in the same way. Then by equation (19), λ_1 and λ_{2N} are periodic eigenvalues of L and thus for any $\lambda_{2n} \le \lambda \le \lambda_{2n+1}$, the left side of equation (A.2) can be estimated by

$$\frac{\sqrt{\Delta(\lambda) - 2}}{\sqrt{\Delta(\lambda) + 2}} = \sqrt{\frac{(\lambda - \lambda_{2n})(\lambda - \lambda_{2n+1})}{(\lambda - \lambda_2)(\lambda - \lambda_{2N-1})}} \cdot R \ge \frac{\sqrt{\lambda - \lambda_{2n}}\sqrt{\lambda_{2n+1} - \lambda}}{\lambda_{2N} - \lambda_1} \cdot R,$$

where

$$R \equiv R(\lambda) = \sqrt{\frac{\lambda - \lambda_1}{\lambda - \lambda_3} \cdots \frac{\lambda - \lambda_{2n-3}}{\lambda - \lambda_{2n-1}} \frac{\lambda_{2n+4} - \lambda}{\lambda_{2n+2} - \lambda} \cdots \frac{\lambda_{2N} - \lambda}{\lambda_{2N-2} - \lambda}}\,. \tag{A.3}$$

As each of the the fractions under the square root in equation (A.3) can be estimated from below by 1, for any $\lambda_{2n} \le \lambda \le \lambda_{2n+1}$ it follows that $R(\lambda) \ge 1$ on $[\lambda_{2n}, \lambda_{2n+1}]$, leading to the claimed estimate (A.2). ∎

B **Appendix Proof of Theorem 3.9**

In this Appendix, we prove Theorem 3.9 using estimates derived in [1]. Let (b, a) be in $\mathcal{M}_{\beta,\alpha}$ with $\beta \in \mathbb{R}$ and $\alpha > 0$ arbitrary.

To show Theorem 3.9, we need the following proposition.

Proposition B.1. For any $(b,a) \in \mathcal{M}_{\beta,\alpha}$ with $\beta \in \mathbb{R}$, $\alpha > 0$ arbitrary and any $1 \le n \le N$,

$$\lambda_{2n}(b,a) - \lambda_{2n-1}(b,a) \le \frac{2\pi\alpha}{N}. \tag{B.1}$$

□

Before proving Proposition B.1 we show how to use it to prove Theorem 3.9.

Proof of Theorem 3.9 We begin by adding up the inequalities (60) and get

$$\sum_{n=1}^{N-1} \gamma_n^2 \le 3\pi\,(\lambda_{2N} - \lambda_1)\left(\sum_{n=1}^{N-1} I_n\right). \tag{B.2}$$

Note that

$$\lambda_{2N} - \lambda_1 = \sum_{n=1}^{N-1} \gamma_n + \sum_{n=1}^{N} (\lambda_{2n} - \lambda_{2n-1}).$$

By the estimate of Proposition B.1 we get for any $(b,a) \in \mathcal{M}_{\beta,\alpha}$,

$$\lambda_{2N} - \lambda_1 \le 2\pi\alpha + \sum_{n=1}^{N-1} \gamma_n,$$

which we substitute into equation (B.2) to yield

$$\sum_{n=1}^{N-1} \gamma_n^2 \le 6\alpha\pi^2 \left(\sum_{n=1}^{N-1} I_n\right) + 3\pi\left(\sum_{n=1}^{N-1} \gamma_n\right)\left(\sum_{n=1}^{N-1} I_n\right).$$

Using the inequality

$$2ab \le \epsilon^2 a^2 + \frac{1}{\epsilon^2} b^2 \quad (a, b \in \mathbb{R}, \epsilon > 0)$$

with $a = \sum_{n=1}^{N-1} \gamma_n$, $b = \sum_{n=1}^{N-1} I_n$, and $\epsilon^2 = \frac{1}{3\pi(N-1)}$, one gets

$$\sum_{n=1}^{N-1} \gamma_n^2 \le 6\pi^2\alpha\left(\sum_{n=1}^{N-1} I_n\right) + \frac{3\pi}{2}\left(\frac{1}{3\pi(N-1)}\left(\sum_{n=1}^{N-1} \gamma_n\right)^2 + 3\pi(N-1)\left(\sum_{n=1}^{N-1} I_n\right)^2\right).$$

As $\left(\sum_{n=1}^{N-1}\gamma_n\right)^2 \le (N-1)\left(\sum_{n=1}^{N-1}\gamma_n^2\right)$, one then concludes that

$$\frac{1}{2}\sum_{n=1}^{N-1}\gamma_n^2 \le 6\pi^2\alpha\left(\sum_{n=1}^{N-1} I_n\right) + \frac{9\pi^2}{2}(N-1)\left(\sum_{n=1}^{N-1} I_n\right)^2, \tag{B.3}$$

which is the claimed estimate (61). ■

To prove Proposition B.1 we first need to make some preparations. Note that for an element of the form $(b,a) = (\beta 1_N, \alpha 1_N)$ one has, by Lemma 2.6,

$$\begin{aligned}\lambda_{2n}(\beta 1_N, \alpha 1_N) - \lambda_{2n-1}(\beta 1_N, \alpha 1_N) &= 2\alpha\left(\cos\frac{(n-1)\pi}{N} - \cos\frac{n\pi}{N}\right)\\ &= 4\alpha\sin\frac{(2n-1)\pi}{2N}\sin\frac{\pi}{2N}\\ &< \frac{2\pi\alpha}{N}.\end{aligned}$$

Hence to prove Proposition B.1, it suffices to show that for any $(b,a) \in \mathcal{M}_{\beta,\alpha}$ and any $1 \le n \le N$,

$$\lambda_{2n}(b,a) - \lambda_{2n-1}(b,a) \le \lambda_{2n}(-\beta 1_N, \alpha 1_N) - \lambda_{2n-1}(-\beta 1_N, \alpha 1_N). \tag{B.4}$$

To this end, following [14] (cf. also [7]), we introduce the conformal map

$$\delta(\lambda) := (-1)^N \int_{\lambda_1}^{\lambda} \frac{\dot\Delta(\mu)}{\sqrt{4-\Delta^2(\mu)}}\, d\mu, \tag{B.5}$$

where the sign of the square root is chosen such that for $\mu < \lambda_1$, $\sqrt{4-\Delta^2(\mu)}$ has positive imaginary part. It is defined on the upper half plane $U := \{\operatorname{Im} z > 0\}$ and its image is the spike domain

$$\Omega(b,a) := \{x+iy : 0 < x < N\pi,\ y > 0\} \setminus \bigcup_{n=1}^{N-1} T_n,$$

where for $1 \le n \le N-1$, T_n denotes the spike

$$T_n := \left\{n\pi + it : 0 < t \le \operatorname{arcosh}\left((-1)^{N+n}\frac{\Delta(\dot\lambda_n)}{2}\right)\right\}.$$

To see that $\delta(U) = \Omega(b, a)$, note that for any $(b, a) \in \mathcal{M}$ and $\lambda \in U$. the discriminant $\Delta(\lambda)$ and the function $\delta(\lambda)$ are related by the formula

$$\Delta(\lambda) = 2(-1)^N \cos \delta(\lambda). \tag{B.6}$$

To prove equation (B.6), recall that for $-1 < t < 1$, one has $\frac{d}{dt} \arccos t = \frac{1}{\sqrt[+]{1-t^2}}$. This formula remains valid for any t in $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Thus

$$\delta(\lambda) = (-1)^N \int_{\lambda_1}^{\lambda} \frac{\dot{\Delta}(\mu)}{\sqrt{4 - \Delta^2(\mu)}}\, d\mu = \int_{\lambda_1}^{\lambda} \frac{d}{d\mu} \arccos\left((-1)^N \frac{\Delta(\mu)}{2}\right).$$

Since by equation (19) $\Delta(\lambda_1) = 2(-1)^N$, we then get

$$\delta(\lambda) = \arccos\left((-1)^N \frac{\Delta(\mu)}{2}\right)\Big|_{\lambda_1}^{\lambda} = \arccos\left((-1)^N \frac{\Delta(\lambda)}{2}\right), \tag{B.7}$$

leading to formula (B.6) and the claimed statement that $\delta(U) = \Omega(b, a)$.

The map δ can be extended continuously to the closure $\{\operatorname{Im} z \geq 0\}$ of the upper half plane. This extension, again denoted by δ, is 2-1 over each nontrivial spike T_n and 1-1 otherwise. Since the nth spike T_n is the image under δ of the nth gap $(\lambda_{2n}, \lambda_{2n+1})$, all spikes are empty if and only if all gaps are collapsed. By Lemma 2.6, all gaps are collapsed for $(b, a) = (-\beta 1_N, \alpha 1_N)$, hence $\Omega(b, a) \subset \Omega(-\beta 1_N, \alpha 1_N)$ for any $(b, a) \in \mathcal{M}_{\beta,\alpha}$.

Note that

$$\lambda_{2n}(b, a) - \lambda_{2n-1}(b, a) = \delta^{-1}(n\pi-) - \delta^{-1}((n-1)\pi+) = \int_{-\infty}^{\infty} u^{(n)}(\delta(\lambda))d\,\lambda, \tag{B.8}$$

where $u^{(n)} : \Omega(b, a) \to \mathbb{R}$, $z \mapsto u^{(n)}(z; b, a)$ is the harmonic measure of the open subset $((n-1)\pi, n\pi)$ of $\partial\Omega(b, a)$ (see e.g. [6] for the notion of the harmonic measure).

We need two lemmas from complex and harmonic analyses, respectively.

Lemma B.1. For $(b, a) = (-\beta 1_N, \alpha 1_N)$ with $\beta \in \mathbb{R}$, $\alpha > 0$ arbitrary, the map $\delta(\lambda)$ defined by equation (B.5) is given by

$$\delta(\lambda) = N \arccos\left(-\frac{\lambda + \beta}{2\alpha}\right). \tag{B.9}$$

For arbitrary (b, a) in $\mathcal{M}_{\beta,\alpha}$ and $\xi \in \mathbb{R}$, the following asymptotic estimate holds as $\eta \to \infty$:

$$\delta(\xi + i\eta) = N \arccos\left(-\frac{\xi + i\eta + \beta}{2\alpha}\right) + O(\eta^{-2}), \tag{B.10}$$

locally uniformly in ξ. □

Proof of Lemma B.1 In view of formulas (51) and (52) for the fundamental solutions y_1 and y_2 for $(b, a) = (-\beta 1_N, \alpha 1_N)$, the discriminant $\Delta(\lambda) = \Delta(\lambda, -\beta 1_N, \alpha 1_N)$ is given by

$$\begin{aligned}\Delta(\lambda) &= y_1(N, \lambda) + y_2(N+1, \lambda)\\ &= -\frac{\sin(\rho(N-1))}{\sin\rho} + \frac{\sin(\rho(N+1))}{\sin\rho}\\ &= 2\cos(\rho N),\end{aligned}$$

where $\pi < \rho < 2\pi$ is determined by $\cos\rho = \frac{\lambda+\beta}{2\alpha}$. Hence

$$\Delta(\lambda) = 2T_N\left(\frac{\lambda+\beta}{2\alpha}\right), \tag{B.11}$$

where for any $z \in U$,

$$T_N(z) = \cos(N \arccos z). \tag{B.12}$$

Actually, $T_N(z)$ is a polynomial in z of degree N, referred to as Chebychev polynomial of the first kind. Substituting equation (B.11) into equation (B.7), we obtain

$$\delta(\lambda) = \arccos\left((-1)^N \frac{\Delta(\lambda)}{2}\right) = \arccos\left((-1)^N T_N\left(\frac{\lambda+\beta}{2\alpha}\right)\right).$$

The claimed identity (B.9) now follows from the elementary symmetry

$$T_N(z) = (-1)^N T_N(-z) \quad \forall z \in \mathbb{C}.$$

Now let $(b, a) \in \mathcal{M}_{\beta,\alpha}$. The asymptotic estimate (B.10) follows by comparing the polynomials $\Delta(\lambda)$ correponding to (b, a) and the one corresponding to $(-\beta 1_N, \alpha 1_N)$. By equation (17) (and the discussion following it), in both cases,

$$\Delta(\lambda) = \alpha^{-N}\lambda^N\left(1 + N\beta\lambda^{-1} + O(\lambda^{-2})\right) \quad \text{as} \quad |\lambda| \to \infty.$$

This implies that

$$\Delta_{b,a}(\lambda) = \Delta_{-\beta 1_N, \alpha 1_N}(\lambda) \cdot (1 + O(\lambda^{-2})),$$

hence by equations (B.9) and (B.11),

$$\begin{aligned}\delta_{b,a}(\lambda) &= \arccos\left((-1)^N \frac{\Delta_{b,a}(\lambda)}{2}\right) \\ &= \arccos\left(\frac{(-1)^N}{2}\Delta_{-\beta 1_N, \alpha 1_N}(\lambda) \cdot (1 + O(\lambda^{-2}))\right). \\ &= \arccos\left((-1)^N T_N\left(\frac{\lambda+\beta}{2\alpha}\right) \cdot (1 + O(\lambda^{-2}))\right) \\ &= \arccos\left(T_N\left(-\frac{\lambda+\beta}{2\alpha}\right) \cdot (1 + O(\lambda^{-2}))\right).\end{aligned}$$

Substituting formula (B.12) for T_N, one then concludes that

$$\begin{aligned}\delta_{b,a}(\lambda) &= \arccos\left(\cos\left(N \arccos\left(-\frac{\lambda+\beta}{2\alpha}\right)\right) \cdot (1 + O(\lambda^{-2}))\right) \\ &= N \arccos\left(-\frac{\lambda+\beta}{2\alpha}\right) + O(\lambda^{-2}),\end{aligned} \tag{B.13}$$

where in the last step we used that $\arccos z = -i \log(z + i \sqrt[+]{1-z^2})$ for any z in $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. ■

Lemma B.3. Let $u : \Omega(b,a) \to \mathbb{R}$ be a bounded harmonic function such that the non-tangential limit of $u(z)$ on $\partial\Omega(b,a)$ has compact support, and let $U(\lambda) := u(\delta(\lambda))$, where $\delta(\lambda)$ is the function defined by equation (B.5). Then for almost every $t \in \mathbb{R}$, the limit $U(t) := \lim_{\eta \to 0} U(t + i\eta)$ exists and is integrable, and

$$\int_{-\infty}^{\infty} U(t) dt = \lim_{x \to \infty} 2\pi\alpha \sinh\left(\frac{x}{N}\right) u\left(\frac{N\pi}{2} + ix\right). \tag{B.14}$$

□

Proof of Lemma B.3 Again by Fatou's theorem, for a.e. t, the (nontangential) limit $\lim_{\eta \to 0} U(t + i\eta)$ exists, since U is a bounded harmonic function on $\{\mathrm{Im}\,(z) > 0\}$. Since u is bounded on $\Omega(b,a)$ and its nontangential limit to $\partial\Omega$ has compact support, $U(t)$ is bounded and compactly supported, and thus in particular, integrable. For $\lambda = \xi + i\eta$, one then has the Poisson representation

$$U(\xi + i\eta) = \frac{\eta}{\pi} \int_{-\infty}^{\infty} \frac{U(t)}{(t-\xi)^2 + \eta^2}, \tag{B.15}$$

and by dominated convergence we conclude that

$$\int_{-\infty}^{\infty} U(t)\, dt = \lim_{\eta \to \infty} \pi \eta\, U(\xi + i\eta). \tag{B.16}$$

(In particular, the limit in the latter expression exists.) In order to compute the right-hand side of equation (B.16), let $\xi + i\eta$ be given by

$$\xi + i\eta = \delta^{-1}\left(\frac{N\pi}{2} + ix\right),$$

for x sufficiently large. Then $U(\xi + i\eta) = u(\frac{N\pi}{2} + ix)$, and by equation (B.13), it follows that

$$\frac{\pi}{2} + \frac{i}{N}x = \arccos\left(-\frac{(\xi+\beta) + i\eta}{2\alpha}\right) + O\left((\xi + i\eta)^{-2}\right) \quad (x \to \infty). \tag{B.17}$$

Taking the cosine of both sides of equation (B.17), multiplying by -2α, and using that $\cos(\frac{\pi}{2} + it) = -i \sinh t$ for $t \in \mathbb{R}$, we obtain

$$2i\alpha \sinh \frac{x}{N} = ((\xi + \beta) + i\eta)(1 + O((\xi + i\eta)^{-2})) \quad (x \to \infty).$$

Hence, as $x \to \infty$,

$$\xi = O(1), \quad \eta = 2\alpha \sinh \frac{x}{N} + O(1). \tag{B.18}$$

Substituting equation (B.18) into equation (B.16) leads to the claimed formula (B.14). ■

Proof of Proposition B.1 Let $(b, a) \in \mathcal{M}_{\beta,\alpha}$. Besides the harmonic measure $u^{(n)}$ of the set $E := ((n-1)\pi, n\pi) \subset \partial\Omega(b, a)$, we also consider the harmonic measure $u^{(n)}_{\beta,\alpha}$ of $E \subset \partial\Omega(-\beta 1_N, \alpha 1_N)$; note that

$$\Omega(-\beta 1_N, \alpha 1_N) = \{x + iy \mid 0 < x < N\pi,\ y > 0\},$$

and hence $\Omega(b, a) \subseteq \Omega(-\beta 1_N, \alpha 1_N)$. According to [6], both $u^{(n)}$ and $u^{(n)}_{\beta,\alpha}$ satisfy the hypotheses of Lemma B.3. Let us recall the monotonicity property of the harmonic measures

$u(z, E, \Omega)$ with respect to Ω (see e.g. [6]): If $\Omega_1 \subseteq \Omega_2$, $E \subset \partial\Omega_1 \cap \partial\Omega_2$, and $u(z, E, \Omega_i)$ $(i = 1, 2)$ denotes the harmonic measure of $E \subset \partial\Omega_i$, then for any $z \in \Omega_1$, $u(z, E, \Omega_1) \leq u(z, E, \Omega_2)$. Apply this general principle to $\Omega_1 := \Omega(b, a)$ and $\Omega_2 := \Omega(-\beta 1_N, \alpha 1_N)$ to get

$$u^{(n)}(x) \leq u^{(n)}_{\beta,\alpha}(x). \tag{B.19}$$

Writing $U^{(n)}(\lambda) := u^{(n)}(\delta(\lambda))$ as well as $U^{(n)}_{\beta,\alpha}(\lambda) := u^{(n)}_{\beta,\alpha}(\delta(\lambda))$ and combining equations (B.8), (B.14), and (B.19), we conclude that

$$\begin{aligned}
\lambda_{2n}(b, a) - \lambda_{2n-1}(b, a) &= \int_{-\infty}^{\infty} U^{(n)}(\lambda) d\,\lambda \\
&= \lim_{x \to \infty} 2\pi\alpha \sinh\left(\frac{x}{N}\right) u^{(n)}\left(\frac{N\pi}{2} + ix\right) \\
&\leq \lim_{x \to \infty} 2\pi\alpha \sinh\left(\frac{x}{N}\right) u^{(n)}_{\beta,\alpha}\left(\frac{N\pi}{2} + ix\right) \\
&= \int_{-\infty}^{\infty} U^{(n)}_{\beta,\alpha}(\lambda) d\,\lambda \\
&= \lambda_{2n}(-\beta 1_N, \alpha 1_N) - \lambda_{2n-1}(-\beta 1_N, \alpha 1_N).
\end{aligned}$$

This completes the proof of estimate (B.1) and therefore of Proposition B.1. ■

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