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Amenable actions of discrete groups

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Abstract. A structure theorem is established for amenable actions of a countable discrete group.

0. *Introduction*

In 1978, R. J. Zimmer introduced the notion of amenability for an action of a separable locally compact group, or an equivalence relation, on a standard Borel space with a probability measure. In **[Z1]**, he showed that the Mackey range of a homomorphism from an amenable ergodic equivalence relation to a separable locally compact group G is an amenable G-space.

In **[CW]**, A. Connes and E. J. Woods studied group-invariant, time-dependent Markov random walks. In particular, they pointed out that the Poisson boundary of a group-invariant matrix-valued random walk on a separable locally compact group is an amenable G-space.

In the case that G is a (countable) discrete group, we shall show in this paper that every amenable ergodic G-space can be obtained by either of these two constructions. To summarize, if (S, μ) is a standard measured space and G acts ergodically on (S, μ) then the following statements are equivalent:

(i) (S, μ) is an amenable G-space;

(ii) the equivalence relation induced by the action of G on (S, μ) is amenable and the stability subgroup $G_s = \{g \in G;\, sg = s\}$ is amenable μ-a.e.;

(iii) (S, μ) is isomorphic as a G-space to the Mackey range of a homomorphism from an amenable equivalence relation to G;

(iv) (S, μ) is isomorphic as a G-space to the Poisson boundary of a group-invariant matrix-valued Markov random walk on G.

The organization of this paper is as follows. In § 1 we shall show that if (S, μ) is an amenable ergodic G-space, then the stabiliser subgroups $G_s = \{g \in G;\, sg = s\}$ are amenable μ-a.e. This fact was stated without proof in **[ST]**; for the sake of completeness (and because we need this result), we prove it here. We shall show in § 2 that (ii) implies (iii), and in § 3 that (iii) implies (iv). In § 4, we shall state the main theorem formally and complete the proof.

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The results of this paper were announced at ICM-90 in Kyoto, where we learned that V. Ya. Golodets and S. D. Sinelshchikov had proved independently and simultaneously the equivalence between (i), (ii) and (iii).

1. *Stabilizers of amenable actions*

Let (S, μ) be a standard measured Borel space, and let G be a countable discrete group, acting ergodically and amenably on (S, μ). By [**Se**], Theorem 1.1, there is a natural short exact sequence of groupoids

$$\Gamma(S\times G)\to S\times G\overset{\pi}{\to}\mathcal{R}_G$$

where $\Gamma(S\times G)=\{(s, g);\, g\in G_s\}$ is the field of isotropy subgroups G_s and $\mathcal{R}_G$ is the equivalence relation induced by G on S. As $\mathcal{R}_G$ is amenable [**Z1**, Proposition 3.4], there exists an ergodic automorphism of (S, μ) such that $\mathcal{R}_G=\mathcal{R}_{\mathbb{Z}}$ a.e. [**CFW**, Proposition 4.1].

It follows that the surjective groupoid map $(s, g)\mapsto(s, sg)$ has a left inverse,

$$\gamma:\mathcal{R}_G\to S\times G,$$

also a (Borel) groupoid map (see [**ST**, pp 1091 to 1092]). Following [**Se**] and [**ST**], let us consider the groupoid semidirect product of $\Gamma(S\times G)$ by $\mathcal{R}_G$ with respect to the action Adγ,

$$\Gamma(S\times G)\rtimes\mathcal{R}_G=\{(s, g_s, sg);\, g_s\in G_s, g\in G\},$$

where

$$r(s, g_s, sg)=s,\quad s(s, g_s, sg)=sg\quad\text{and}$$

$$(s, g_s, sg)(sg, g_{sg}, sgh)=(s, g_s\gamma(s, sg)g_{sg}\gamma^{-1}(s, sg), sgh).$$

This groupoid is endowed with the measure

$$\nu(\cdot)=\int_S |r^{-1}(s)\cap\cdot|\, d\mu(s).$$

Clearly, $\Gamma(S\times G)\rtimes\mathcal{R}_G$ is isomorphic to $S\times G$ by the map

$$(s, g_s, sg)\in\Gamma(S\times G)\rtimes\mathcal{R}_G\mapsto(s, g_s\gamma(s, sg))\in S\times G.$$

Let us recall (see [**H1**]) the definition of the von Neumann algebra of the groupoid $\Gamma(S\times G)\rtimes\mathcal{R}_G$. Denote the set $\{f\in L^1(\Gamma(S\times G)\rtimes\mathcal{R}_G);\, \|f\|_\Pi<\infty\}$ by $\Pi(\Gamma(S\times G)\rtimes\mathcal{R}_G)$, where the norm $\|f\|_\Pi$ is equal to

$$\sup\left\{\int |f(s, g_s, sg)\xi(s)\eta(sg)|\, d\nu(s, g_s, sg);\, \xi, \eta\in L^2(S, \mu), \|\xi\|=\|\eta\|=1\right\}.$$

For each $f\in\Pi(\Gamma(S\times G)\rtimes\mathcal{R}_G)$, the map L_f which to each function ξ belonging to $L^2(\Gamma(S\times G)\rtimes\mathcal{R}_G)$ assigns the function

$$L_f(\xi)(x)=\int f(xy)\xi(y^{-1})\, d\nu^{s(x)}(y),\quad x\in\Gamma(S\times G)\rtimes\mathcal{R}_G,$$

is a bounded operator on $L^2(\Gamma(S\times G)\rtimes\mathcal{R}_G)$.

The von Neumann algebra $W^*(\Gamma(S\times G)\rtimes\mathcal{R}_G)$ of the groupoid $\Gamma(S\times G)\rtimes\mathcal{R}_G$ is the von Neumann algebra on $L^2(\Gamma(S\times G)\rtimes\mathcal{R}_G)$ generated by the L_f, $f\in\Pi(\Gamma(S\times G)\rtimes\mathcal{R}_G)$.

If $f\in\Pi(\Gamma(S\times G)\rtimes\mathcal{R}_G)$, let us denote by $E(f)$ the restriction of f to the 'diagonal', i.e.,

$$E(f)(s, g_s, sg)=\begin{cases}0 & \text{if } s\neq sg,\\ f(s, g_s, s) & \text{if } s=sg.\end{cases}$$

For $\xi\in L^2(\Gamma(S\times G)\rtimes\mathcal{R}_G)$ and $f\in\Pi(\Gamma(S\times G)\rtimes\mathcal{R}_G)$, we have:

$$\begin{aligned}
L_{E(f)}\xi(s, g_s, t) &= \int f((s, g_s, t)(t, g_t, s))\xi(s, \gamma(s, t)g_t^{-1}\gamma(t, s), t)\,d\nu^t(t, g_t, s)\\
&= \int f(s, g_s\gamma(s, t)g_t\gamma(t, s), s)\xi(s, \gamma(s, t)g_t^{-1}\gamma(t, s), t)\,d\nu^t(t, g_t, s)\\
&= \sum_{G_t} f(s, g_s\gamma(s, t)g_t\gamma(t, s), s)\xi(s, \gamma(s, t)g_t^{-1}\gamma(t, s), t)\\
&= \sum_{G_s} f(s, g_s h_s^{-1}, s)\xi(s, h_s, t)\\
&= \sum_{G_s} f(s, k_s, s)\xi(s, k_s^{-1}g_s, t).
\end{aligned}$$

One deduces easily the following

LEMMA 1.1. *Let N denote the sub von Neumann algebra of $W^*(\Gamma(S\times G)\rtimes\mathcal{R}_G)$ generated by $\{L_{E(f)}; f\in\Pi(\Gamma(S\times G)\rtimes\mathcal{R}_G)\}$.*

(i) *N is isomorphic to $\int_S^\oplus \lambda(G_s)''\,d\mu(s)$.*

(ii) *The map $E(L_f)=L_{E(f)}$ extends to a normal conditional expectation from $W^*(\Gamma(S\times G)\rtimes\mathcal{R}_G)$ onto N.*

PROPOSITION 1.2. *Let (S,μ) and G be as above. The stability subgroup $G_s=\{g\in G; sg=s\}$ is amenable a.e.*

Proof. As $S\times G\cong\Gamma(S\times G)\rtimes\mathcal{R}_G$ and the von Neumann algebra of a (measured) groupoid is an isomorphism invariant, the factor $W^*(\Gamma(S\times G)\rtimes\mathcal{R}_G)$ is injective by [**Z2**], Theorem 2.1.

Hence by Lemma 1.1, the algebra $\int_S^\oplus \lambda(G_s)''\,d\mu(s)$ is injective and by [**C1**, Proposition 6.5], $\lambda(G_s)''$ is injective a.e.

Therefore by the result of J. T. Schwartz, G_s is amenable a.e. □

2. *Amenable actions and Mackey ranges*

In this paragraph, we shall prove the following

THEOREM 2.1. *Let S be a standard Borel space with a non-atomic probability measure μ and let G be a (countable) discrete group acting ergodically on S with respect to μ. Assume that*

(i) *the equivalence relation induced by G on S is amenable, and*

(ii) *the stability subgroup $G_s=\{g\in G; sg=s\}$ is amenable, for μ-almost every $s\in S$.*

Then there exist a standard measured Borel space $(\tilde{S}, \tilde{\mu})$, *an amenable, discrete equivalence relation* $\tilde{\mathcal{R}}$ *on* $(\tilde{S}, \tilde{\mu})$, *and a cocycle* $\alpha: \tilde{\mathcal{R}} \to G$ *such that the Mackey range of* α *is isomorphic as a G-space to* (S, μ).

The proof of this theorem divides into three parts. Note first that the space

$$\tilde{S} = \{(s, (g_n^s)_{n \geq 1}); g_n^s \in G_s \text{ for all } n \geq 1\} \subset S = \prod_{n \geq 1} G$$

is a standard Borel space (Lemma 2.9). In fact the first part of the proof, from Lemma 2.2 to Proposition 2.10, a measure $\tilde{\mu}$ on $\tilde{S}$ is built. In the second part of the proof, from Lemma 2.11 to Lemma 2.13, we define an equivalence relation $\tilde{R}$ on $(\tilde{S}, \tilde{\mu})$, and show that $\tilde{R}$ is discrete, non-singular, and amenable. In the last part of the proof, we construct a homomorphism $\alpha: \tilde{\mathcal{R}} \to G$, and a G-isomorphism between the Mackey range of α and the G-space (S, μ).

Let S be a standard Borel space and G a countable group. Denote by $\mathcal{P}f(G)$ the collection of all finite subsets of G.

As G is countable, $\mathcal{P}f(G)$ is countable, and so with the discrete topology it is a Polish space.

Let $\mathrm{S}(G)$ be the space of subgroups of G, with the standard Borel structure generated by the sets $\{K \in \mathrm{S}(G); K \subseteq A\}$, where A runs through the subsets of G (these form the closed sets for a Polish topology on $\mathrm{S}(G)$).

LEMMA 2.2. *Let* $s \in S \mapsto G_s \in \mathrm{S}(G)$ *be a Borel map.*
Then the subset $E = \{(s, F); F \subset G_s\}$ *of* $S \times \mathcal{P}f(G)$ *is Borel.*

Proof. The subset $\{(s, g); g \in G_s\}$ of $S \times G$ is Borel (see for example [**S1**], Theorem 2.5). Therefore, with $G = \{g_0, g_1, \ldots\}$, for each $n \geq 0$, the sets $S_n = \{s \in S; g_n \in G_s\}$ and $S_n^c = \{s \in S; g_n \notin G_s\}$ are Borel. Denote by π the projection of $S \times \mathcal{P}f(G)$ onto S. For each $n \geq 0$, the set $A_n = \pi^{-1}(S_n) \cup (\pi^{-1}(S_n^c) \cap S \times \mathcal{P}f(G \backslash \{g_n\})$ is a Borel subset of $S \times \mathcal{P}f(G)$, and $E \subset A_n$. Conversely, if $(s, F) \in \bigcap_{n \geq 0} A_n$, then for each $n \geq 0$,

$$g_n \in F \;\Rightarrow\; g_n \in G_s,$$

and so $(s, F) \in E$. Therefore, $E = \bigcap_{n \geq 0} A_n$, and the lemma is proved. □

Now let G act on S and suppose that S is a Polish G-space under the action $(s, g) \in S \times G \mapsto sg \in S$.

For all $s \in S$, set $G_s = \{g \in G; sg = s\}$. Then $s \in S \mapsto G_s \in S(G)$ is a Borel map.

Let μ be a G-quasi-invariant probability measure on S and assume that G_s is amenable μ-a.e. Then

LEMMA 2.3. *For each* $n \geq 1$ *and* $\delta > 0$, *the subset*

$$E_{n,\delta} = \left\{(s, F); F \subset G_s, \frac{|g_i F \triangle F|}{|F|} < \delta \text{ if } g_i \in G_s, i = 1, \ldots, n\right\}$$

of $S \times \mathcal{P}f(G)$ *is Borel. Moreover,* μ (Domain $E_{n,\delta}) = 1$.

Proof. If π denotes the projection $S \times \mathcal{P}f(G) \to S$, then by definition

$$\text{Domain}\,(E_{n,\delta}) = \{s \in S; \pi^{-1}(s) \cap E_{n,\delta} \neq \emptyset\}.$$

First, note that for all $g \in G$, the map

$$F \in \mathscr{P}f(G) \mapsto \frac{|gF \triangle F|}{|F|} \in \mathbb{Q} \quad \text{is Borel.}$$

As $s \mapsto G_s$ is Borel, the set $S_k = \{s \in S;\ g_k \in G_s\}$ is Borel for $k \geq 0$. Then for each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n$, the subset

$$S_\varepsilon = S_{\varepsilon_1,\ldots,\varepsilon_n} = \{s \in S;\ g_i \in G_s \text{ if } \varepsilon_i = 1,\ g_i \notin G_s \text{ if } \varepsilon_i = 0\}$$

$$= \Big(\bigcap_{\varepsilon_i = 1} S_i\Big) \cap \Big(\bigcap_{\varepsilon_i = 0} S_i^c\Big)$$

is Borel, and $S = \coprod_{\varepsilon \in \{0,1\}^n} S_\varepsilon$. As

$$E_{n,\delta} = \coprod_{\varepsilon \in \{0,1\}} \{(s, F);\ F \subset G_s\} \cap S \times \left\{F;\ \frac{|g_{i\varepsilon_i} F \triangle F|}{|F|} < \delta\right\} \cap \pi^{-1}(S_\varepsilon),$$

$E_{n,\delta}$ is a Borel subset of $S \times \mathscr{P}f(G)$ by Lemma 2.2.

Moreover, as G_s is amenable μ-a.e., $\pi^{-1}(s) \cap E_{n,\delta} \neq \emptyset$ μ-a.e. □

Let us denote by $\mathscr{G}$ the set $\{\bar{g} : S \to G;\ \bar{g}$ is a Borel map and $\bar{g}_s \in G_s\}$. If ϕ is a map from S to the set $\mathscr{F}(G)$ of functions on G and $\bar{g} \in \mathscr{G}$, then the map $\bar{g}\phi : S \to \mathscr{F}(G)$ is defined by

$$\bar{g}\phi(s) = \phi(s)(\bar{g}(s)^{-1}h), \quad s \in S, h \in G.$$

Then by [**A**, Theorem 2.4] (for example), together with Lemma 2.3, there exists a Borel map

$$F\colon \text{Domain}\,(E_{n,\delta}) \subset S \to \mathscr{P}f(G)$$
$$s \mapsto F_s$$

the graph of which is contained in $E_{n,\delta}$. Set $\varphi_{n,\delta}(s) = \chi_{Fs}/|F_s|$, $s \in \text{Domain}\,(E_{n,\delta})$. Then $\varphi_{n,\delta}(s) \in l^1(G_s)_+$, $\|\varphi_{n,\delta}(s)\|_1 = 1$, and the map

$$\varphi_{n,\delta}\colon \text{Domain}\,(E_{n,\delta}) \subset S \to l^1(G)$$
$$s \mapsto \varphi_{n,\delta}(s)$$

is Borel, and $\|g_i\varphi_{n,\delta}(s) - \varphi_{n,\delta}(s)\|_1 < \delta$ if $g_i \in G_s$, $i = 1, \ldots, n$, for a.e. $s \in S$.

From this we deduce the following

LEMMA 2.4. *There exists a sequence $(\varphi_n)_{n\geq 1}$ of Borel functions from S to $l^1(G)$ such that*

(a) $\varphi_n(s) \in l^1(G_s)_+$, $\|\varphi_n(s)\|_1 = 1$, *and*

(b) *for any* $\bar{g} \in \mathscr{G}$,

$$\|\bar{g}(s)\varphi_n(s) - \varphi_n(s)\|_1 \to 0 \quad \text{a.e.}$$

Proof. Let $\varphi_{n,\delta}$ be as above. Then $D = \bigcap_{n\geq 1} \text{Domain}\,(E_{n,n^{-1}})$ is conull in S. Define $\varphi_n : S \to l^1(G)$ by

$$\varphi_n(s) = \begin{cases} \varphi_{n,n^{-1}}(s) & \text{if } S \in D, \\ \chi_{\{e\}} & \text{if } S \notin D. \end{cases}$$

We just have to check (b).

Let $\bar{g} \in \mathcal{G}$. Then let $(S_n(\bar{g}))_{n\geq 0}$ denote the Borel partition of S, where $S_n(\bar{g})$ is the set $\{s \in S;\ \bar{g}(s) = g_n\}$. For each N, for each $k \geq N+1$,

$$\|\bar{g}\varphi_k(s) - \varphi_k(s)\|_1 \leq \frac{1}{N+1} \quad \text{for a.e. } s \in \coprod_{n=1}^{k} S_n(\bar{g}).$$

Hence,

$$\mu\left(\bigcup_{m\geq k+1} \{s \in S;\ \|\bar{g}(s)\varphi_m(s) - \varphi_m(s)\|_1 \geq N^{-1}\}\right) \leq \sum_{m\geq k+1} \mu(S_m(\bar{g})).$$

Therefore (see, for example, Theorem 1 on page 251 of [**Sh**]),

$$\|\bar{g}\varphi_n(s) - \varphi_n(s)\|_1 \to 0 \quad \text{a.e.} \qquad \square$$

Let E denote the Banach space $L^1(S, l^\infty(G))$ (with $l^\infty_{\mathbb{R}}(G)$). We define an order structure on E by

$$f \geq 0 \quad \text{if} \quad f(s) \geq 0 \quad \text{in } l^\infty(G), \text{a.e.}$$

Consider the two elements ι and c of E defined by

$$\iota(s)(g) = 1 \quad \text{for all } s \in S,\ g \in G,$$

$$c(s) = \chi_{G_s} \quad \text{for all } s \in S.$$

We will denote by $\mathcal{M}_E$ the set of means of E, i.e.,

$$\mathcal{M}_E = \{\varphi \in E_1^*;\ \varphi \geq 0 \quad \text{and} \quad \langle \varphi, \iota\rangle = 1\},$$

and we set $\mathcal{M}_{E,c} = \mathcal{M}_E \cap \{\varphi \in E^*;\ \langle\varphi, c\rangle = 1\}$.

Recall that $P^1(G) = \{f \in L^1(G)_+;\ \|f\|_1 = 1\}$. Set

$$\tilde{P}(S) = \{\eta \in L^\infty(S, l^1(G));\ \eta(s) \geq 0 \quad \text{and} \quad \|\eta(s)\|_1 = 1 \text{ a.e.}\}$$

and

$$P(S) = \tilde{P}(S) \cap \{\eta \in L^\infty(S, l^1(G));\ \text{supp}\ \eta(s) = G_s \text{ a.e.}\}.$$

We will need the following results, the proof of which we give for the sake of completeness.

LEMMA 2.5. (*a*) *The space* $L^\infty(S, l^1(G))$ *is isometrically embedded in* E^*, *via the map* $\eta \mapsto \hat{\eta}$ *with*

$$\langle \hat{\eta}, f\rangle = \int_S \langle \eta(s), f(s)\rangle\, d\mu(s), \quad f \in E.$$

(b) *The set* $\mathcal{M}_E$ *is weak*-compact and* $\tilde{P}(S)^\wedge$ *is weak*-dense in* $\mathcal{M}_E$.

(c) *The set* $\mathcal{M}_{E,c}$ *is weak*-compact and* $\hat{P}(S)$ *is weak*-dense in* $\mathcal{M}_{E,c}$.

Proof. (a) Let $\eta \in L^\infty(S, l^1(G))$ and $f \in E$. We have

$$\begin{aligned} |\langle\hat{\eta}, f\rangle| &\leq \int_S |\langle \eta(s), f(s)\rangle|\, d\mu(s) \\ &\leq \int_S \|\eta(s)\|_1 \|f(s)\|_\infty\, d\mu(s) \\ &\leq \|\eta\|\, \|f\|. \end{aligned}$$

Hence, $\|\hat{\eta}\| \leq \|\eta\|$. To verify the other inequality, let us show that for all $\varepsilon > 0$,

$$\|\hat{\eta}\| \geq \|\eta\| - \varepsilon.$$

As (S, μ) is a standard Borel space, with non-atomic measure, we may assume that (S, μ) is isomorphic to $[0, 1]$, with Lebesgue measure. Set $C = \{s \in S; \|\eta(s)\|_1 \geq \|\eta\| - \frac{1}{3}\varepsilon\}$. Then C is measurable with non-zero measure. As η is measurable, there exists a compact set $K \subset C$ such that $\mu(C \backslash K) < \mu(C)/2$ and $\eta | K$ is continuous. Hence there exist $s_1, \ldots, s_n \in K$ and strictly positive real numbers $\delta_1, \ldots, \delta_n$ such that $K \subset \bigcup_{i=1}^n B(s_i, \delta_i)$, and

$$\|\eta(t) - \eta(s_i)\|_1 < \tfrac{1}{3}\varepsilon \quad \text{if } t \in K \cap B(s_i, \delta_i), \text{ for } i = 1, \ldots, n.$$

Then there is a measurable partition $(A_i)_{i=1,\ldots,n}$ of K such that

$$\|\eta(t) - \eta(s_i)\|_1 \leq \tfrac{1}{3}\varepsilon \quad \text{if } t \in A_i.$$

Let $\eta_1 \in L^\infty(S, l^1(G))$ be defined by

$$\eta_1(s) = \begin{cases} \eta(s_i) & \text{if } s \in A_i, \\ \eta(s) & \text{if } s \notin K. \end{cases}$$

Then

$$\|\eta - \eta_1\| = \sup_i \sup_{t \in A_i} \|\eta(t) - \eta(s_i)\|_1 \leq \tfrac{1}{3}\varepsilon, \tag{2.5.1}$$

$$\begin{aligned} |\langle \hat{\eta} - \hat{\eta}_1, f \rangle| &\leq \int_S |\langle \eta(s) - \eta_1(s), f \rangle| \, d\mu(s) \\ &\leq \int_K \|\eta(s) - \eta_1(s)\|_1 \|f(s)\|_\infty \, d\mu(s) \qquad (2.5.1) \\ &\leq \tfrac{1}{3}\varepsilon \|f\|. \end{aligned}$$

For $i = 1, \ldots, n$, set $B_i = \{g \in G; \eta(s_i)(g) \geq 0\} \subset G$. Let $f: S \to l^\infty(G)$ be the map defined by

$$f(s) = \begin{cases} 0 & \text{if } s \notin K, \\ 2\chi_{B_i} - 1 & \text{if } s \in A_i, \, i = 1, \ldots, n. \end{cases}$$

Then $f \in E$ and

$$\begin{aligned} \langle \hat{\eta}_1, f \rangle = \sum_{i=1}^n \int_{A_i} \langle \eta(s_i), 2\chi_{B_i} - 1 \rangle \, d\mu(s) &= \int_K \|\eta_1(s)\|_1 \, d\mu(s) \\ \geq \int_K (\|\eta(s)\| - \tfrac{1}{3}\varepsilon) \, d\mu(s) &\geq (\|\eta\| - \tfrac{2}{3}\varepsilon)\|f\|. \end{aligned} \tag{2.5.2}$$

Here we have used (2.5.1) and the definition of C. By (2.5.1) again together with (2.5.2), we have

$$\|\hat{\eta}\| \, \|f\| \geq |\langle \hat{\eta}, f \rangle| \geq |\langle \hat{\eta}_1, f \rangle| - \tfrac{1}{3}\varepsilon \|f\| \geq (\|\eta\| - \varepsilon)\|f\|.$$

As $\|f\| > 0$, this shows that $\|\hat{\eta}\| \geq \|\eta\| - \varepsilon$.

(b) Note first that $\mathcal{M}_E$ is a weak*-compact subset of E_1^* and that $\tilde{P}(S)$ is a convex subset of $\mathcal{M}_E$. If the weak*-closure of $\tilde{P}(S)$ is not equal to $\mathcal{M}_E$, then by the Hahn–Banach theorem, for some $\varphi \in \mathcal{M}_E$, there exists $f \in E$ such that

$$\langle \varphi, f \rangle > \sup \{\langle \hat{\eta}, f \rangle; \eta \in \tilde{P}(S)\}.$$

We may assume that $f \geq 0$. (By [E, (8.18.1)], we may assume that f is bounded; we may then add a suitable multiple of ι to f.) We will obtain a contradiction, by proving that

$$\sup \{\langle \hat{\eta}, f \rangle;\ \eta \in \tilde{P}(S)\} = \|f\|.$$

Indeed, the simple functions $\sum_{k=1}^{n} \chi_{A_k} f_k$, where $(A_k)_{k=1,\dots,n}$ is a measurable partition of S and $f_k \in l^\infty(G)_+$, are dense in E_+ (see [E, (8.18.1)]). Let $\varepsilon > 0$ be given and choose $\tilde{f} = \sum_{k=1}^{n} \chi_{A_k} f_k$ such that $\|f - \tilde{f}\| < \frac{1}{3}\varepsilon$. For $1 \leq k \leq n$, let $\eta_k \in P^1(G)$ be such that

$$\langle \eta_k, f_k \rangle \geq \|f_k\|_\infty - \tfrac{1}{3}\varepsilon$$

and set $\eta = \sum_{k=1}^{n} \chi_{A_k} \eta_k$. Then $\eta \in \tilde{P}(S)$ and

$$\langle \hat{\eta}, f \rangle \geq \langle \hat{n}, \tilde{f} \rangle - \|f - \tilde{f}\| \geq \|\tilde{f}\| - \tfrac{2}{3}\varepsilon \geq \|f\| - \varepsilon.$$

(c) Clearly, $\mathcal{M}_{E,c}$ is weak*-compact and $P(S) \subset \mathcal{M}_{E,c}$. Let $\varphi \in \mathcal{M}_{E,c}$, $h_1, \dots, h_n \in E$, and $\varepsilon > 0$ be given. By [E, (8.18.1)], there exist $f_1, \dots, f_n \in E$ and $K > 0$ such that

$$\|f_i - h_i\| < \tfrac{1}{4}\varepsilon \quad \text{and} \quad \|f_i(s)\|_\infty \leq K, \quad s \in S. \tag{2.5.3}$$

Let $\varepsilon_1 \leq \varepsilon/2(1+3K)$. By (b), there exists $\eta \in \tilde{P}(S)$ such that

$$|\langle \hat{\eta}, c \rangle - 1| < \varepsilon_1 \ , \quad \text{and} \quad |\langle \hat{\eta}, f_i \rangle - \langle \varphi, f_i \rangle| < \varepsilon_1, \quad i = 1, \dots, n. \tag{2.5.4}$$

For $s \in S$, set $\bar{\eta}(s) = \|\chi_{G_s} \eta(s)\|_1$. By (2.5.4),

$$\int_S (1 - \bar{\eta}(s))\, d\mu(s) \leq \varepsilon_1,$$

and so if $C = \{s \in S;\ \bar{\eta}(s) = 0\}$, then $\mu(C) < 2\varepsilon_1$.

Let $\nu : S \to l^1(G)$ be defined by

$$\nu(s) = \begin{cases} [1/\bar{\eta}(s)]\chi_{G_s}\eta(s) & \text{if } s \notin C, \\ \chi_{\{e\}} & \text{if } s \in C. \end{cases}$$

By construction, $\nu \in P(S)$. We have

$$\begin{aligned} \int_{S \setminus C} |\langle \nu(s) - \eta(s), f_i(s) \rangle|\, d\mu(s) &\leq \int_{S\setminus C} \|f_i(s)\|_\infty 2(1 - \bar{\eta}(s))\, d\mu(s) \leq K\varepsilon_1 \\ &\leq \int_{S \setminus C} \|f_i(s)\|_\infty \left\| \frac{\chi_{G_s} - \bar{\eta}(s)}{\bar{\eta}(s)} \eta(s) \right\|_1 d\mu(s) \end{aligned}$$

and

$$\begin{aligned} \int_C |\langle \nu(s) - \eta(s), f_i(s) \rangle|\, d\mu(s) &\leq \int_C (\|\bar{\nu}(s)\|_1 + \|\eta(s)\|_1) \|f_i(s)\|_\infty\, d\mu(s) \\ &\leq 2K\mu(C) \leq 4K\varepsilon_1 \end{aligned}$$

for $i = 1, \dots, n$. Hence,

$$|\langle \hat{\nu} - \hat{\eta}, f_i \rangle| \leq 3K\varepsilon_1, \quad i = 1, \dots, n. \tag{2.5.5}$$

By (2.5.3), (2.5.4) and (2.5.5), we have

$$|\langle \hat{\nu} - \hat{\eta}, h_i \rangle| \leq \tfrac{1}{2}\varepsilon + |\langle \hat{\nu} - \hat{\eta}, f_i \rangle| \leq \tfrac{1}{2}\varepsilon + \varepsilon_1 + 3K\varepsilon_1 \leq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \leq \varepsilon.$$

Therefore, (c) is proved. □

Denote by $r: S\times G\to \mathbb{R}_+^*$ the Radon–Nikodym cocycle of the action of G on (S,μ). As in § 1, there exists a Borel left inverse $\gamma:\mathcal{R}_G\to S\times G$ of the surjection $S\times G\to\mathcal{R}_G$ with $\gamma((s,sg)(sg,sgh))=\gamma(s,sg)\gamma(sg,sgh)$ on $\mathcal{R}_G^{(2)}$, i.e., such that γ is multiplicative. Let us still denote by γ the composition of γ with the projection from $S\times G$ to G.

Let $\chi: G\to \operatorname{Iso}(E)$ denote the action of G given by

(1) $$\chi_g f(s)=r(s,sg)f(sg)(\gamma(s,sg)^{-1}k\gamma(s,sg)),\quad s\in S, k\in G.$$

If π denotes the action by conjugation on $l^p(G)$ (i.e., $\pi_g f(k)=f(g^{-1}kg)$, $k\in G$, $f\in l^p(G)$), then

(2) $$\chi_g f(s)=r(s,sg)\pi_{\gamma(s,sg)}f(sg),\quad f\in E.$$

For each $\bar g\in\mathcal{G}=\{\bar g: S\to G;\ \bar g \text{ a Borel map with } \bar g(s)\in G_s\}$, there corresponds an isometric automorphism of E, $f\mapsto \bar g f$, defined by

(3) $$\bar g f(s)(k)=f(s)(\bar g(s)^{-1}k),\quad s\in S, k\in G.$$

For all $\bar g\in\mathcal{G}$, $h\in G$, let $\bar g h\in\mathcal{G}$ be defined by

(4) $$\bar g h(s)=\gamma(s,sh)\bar g(sh)\gamma(s,sh)^{-1},\quad s\in S.$$

Then for all $\bar g\in\mathcal{G}$, $h\in G$,

(5) $$\chi_h(\bar g)=(\bar g h)\chi_h.$$

Denote by χ^* the dual action, defined by

(6) $$\langle\chi_g^*\varphi,f\rangle=\langle\varphi,\chi_g^{-1}f\rangle,\quad \varphi\in E^*, f\in E, g\in G,$$

and for each $\bar g\in\mathcal{G}$, denote by $\bar g^*$ the dual transformation

$$\langle\bar g^*\varphi,f\rangle=\langle\varphi,\bar g^{-1}f\rangle,\quad \varphi\in E^*, f\in E.$$

Let ψ be a weak*-cluster point (in E_1^*) of the sequence $(\hat\varphi_n)_{n\ge1}$ (of Lemma 2.4). If $\bar g\in\mathcal{G}$, then $(\bar g^*\hat\varphi_n-\hat\varphi_n)$ converges weak* to 0 by Lemma 2.4. Therefore, ψ is $\mathcal{G}$-invariant. Moreover, $\langle\psi,\iota\rangle=1$, because $\langle\hat\varphi_n,\iota\rangle=1$ for all $n\ge1$.

For all $b\in L^\infty_{\mathbb{R}}(S)$ and $f\in E$, note that bf, defined by $(bf)(s)=b(s)f(s)$, $s\in S$, is in E. Hence, if $\varphi\in E^*$, then $b\varphi$ given by

$$\langle b\varphi,f\rangle=\langle\varphi,bf\rangle,\quad f\in E,$$

is an element of E^*, with $\|b\varphi\|\le\|b\|_\infty\|\varphi\|$. Set

$$\tilde K=\{f:[0,1]\to[0,1];\ f=\textstyle\sum q_i\chi_{A_i}\},$$

where $q_i\in\mathbb{Q}_+$ and $\{A_i\}$ is a finite partition of $[0,1]$ into intervals with rational endpoints. Let $\beta: S\to[0,1]$ be a Borel isomorphism (S is a standard Borel space), and set $K=\{f\beta g; f\in\tilde K, g\in G\}$. Then K is countable.

LEMMA 2.6. *Let A denote the weak*-closure of $\{\sum_{\text{finite}} f_g\chi_g^*(\psi); f_g\in K, \sum_{\text{finite}} f_g=1\}$. Then A is a non-empty, compact, convex subset of $\mathcal{M}_{E,c}$ which is $\chi^*(G)$-invariant and each element of which is $\mathcal{G}$-invariant.*

Proof. As $\psi\in\mathcal{M}_{E,c}$, the first part of the lemma is clear.

Since ψ is $\mathcal{G}$-invariant, by (5), $\chi_h^*\psi$ is also $\mathcal{G}$-invariant for all $h\in G$. Therefore every element of A is $\mathcal{G}$-invariant.

If $f \in L^\infty(S)$, $k \in E$, and $h \in G$, then $\chi_h(fk)(s) = f(sh)\chi_h k(s)$, $s \in S$. If $f \in K$, then $fg \in K$ for each $g \in G$. It follows that A is $\chi^*(G)$-invariant. □

Denote by λ the left regular representation of G. Let F be a $\lambda(G)$-invariant and $\pi(G)$-invariant separable closed subspace of $l^\infty(G)$, containing the constant functions. Denote by $r: l^\infty(G)^* \to F^*$ the restriction map. Then r also defines a map from E^* to $L^1(S, F)^* \cong L^\infty(S, F^*)$. If $f \in L^1(S, F)$, $b \in G$ and $\bar{g} \in \mathcal{G}$, then $\chi_h f$ and $\bar{g}f$ belong to $L^1(S, F)$. Hence r is $\mathcal{G}$- and $\chi^*(G)$-equivariant.

In $\bigoplus_{g\in G}\mathbb{Q}$, set $Q = \{(y_g)_{g\in G};\ y_g \geq 0,\ \sum_{g\in G} y_g = 1\}$. For each $y \in Q$, we denote by $s \mapsto y(r(\psi))(s)$ the element of $L^\infty(S, F_1^*)$ corresponding to

$$y(r(\psi)) = \sum_{g\in G} y_g \chi_g^*(r(\psi)) \in L^1(S, F)_1^*.$$

As $y(r(\psi)) \in \mathcal{M}_E$, it follows that for a.e. $s \in S$, $y(r(\psi))(s)$ is a mean on F (i.e., belongs to $\mathcal{M}_F$).

Let Iso (F) denote the group of isometric isomorphisms of F, endowed with the strong operator topology, and $H(F_1^*)$ the group of homeomorphisms of F_1^*, with the topology of uniform convergence. Consider the (Borel) homomorphism $\pi: \mathcal{R}_G \to$ Iso (F) defined by

$$\pi_{(s,t)} f = \pi_{\gamma(s,t)} f, \quad f \in F \subset l^\infty(G),$$

and denote by $\pi^*: \mathcal{R}_G \to H(F_1^*)$ the induced adjoint homomorphism, given by $\pi^*(s, t) = (\pi(s, t)^*)^{-1}$.

Lemma 2.7. *For every $s \in S$, denote by $A_{F,s}$ the weak* closure of the set*

$$\{y(r(\psi))(s);\ y \in Q\} \subset F_1^*.$$

Then $(A_{F,s})_{s\in S}$ is a Borel field of compact convex subsets of F_1^* such that

$$\pi^*(s, t)A_{F,t} = A_{F,s} \quad a.e.$$

Moreover, $r(A) = \{\varphi \in L^\infty(S, F^*);\ \varphi(s) \in A_{F,s}$ a.e.$\}$.

Proof. By [**Z1**], Lemma 1.7, the map $s \mapsto A_{F,s}$ is a Borel field of compact convex sets of F_1^*. For all $g \in G$ and $k \in L^1(S, F)$, we have

$$\langle \chi_g^* r(\psi), k\rangle = \int_S \langle r(\psi)(s), r(s, sg^{-1})\pi_{\gamma(s,sg^{-1})}k(sg^{-1})\rangle\, d\mu(s)$$

$$= \int_S \langle \pi^*_{\gamma(t,tg)} r(\psi)(tg), k(t)\rangle\, d\mu(t).$$

Hence $s \mapsto \pi^*_{(s,sg)} r(\psi)(sg)$ is the element of $L^\infty(S, F^*)$ corresponding to $\chi_g^* r(\psi)$, and

$$s \mapsto y(r(\psi))(s) = \sum_g y_g \pi^*_{(s,sg)} r(\psi)(sg) \in A_{F,s}.$$

If $(s, t) = (s, sh)$, then

$$\pi^*_{(s,t)} y(r(\psi))(t) = \sum_g y_g \pi^*_{(s,sh)} \pi^*_{(sh,shg)} r(\psi)(shg)$$

$$= \sum_g y_g \pi^*_{(s,shg)} r(\psi)(shg) \in A_{F,s}.$$

Set $A_0=\{\sum_{\text{finite}} f_g\chi_g^*(\psi);\ f_g\in K,\ \sum_{\text{finite}} f_g(s)=1\}\subset E_1^*$ and $B=\{\varphi\in L^\infty(S, F_1^*);\ \varphi(s)\in A_{F,s}$ a.e.$\}$. By [**Z1**], Proposition 2.2, B is a closed convex subset of $L^\infty(S, F^*)_1$. As r is continuous and A is compact, the lemma will be proved if we show that $r(A_0)$ is weak*-dense in B. By [**Z1**], Lemma 2.5, functions of the form

$$s\mapsto\sum_{i=1}^n f_i(s)y_i(r(\psi))(s)$$

are dense in B. But we have

$$\sum_{i=1}^n f_iy_i(\psi)=\sum_g\left(\sum_{i=1}^n y_{i,g}f_i\right)\chi_g^*(\psi)\in A_0,$$

and $r(\sum_{i=1}^n f_iy_i(\psi))$ corresponds to $s\mapsto\sum_{i=1}^n f_i(s)y_i(r(\psi))(s)$. □

LEMMA 2.8. *With the above notation, if $\mathcal{R}_G$ is an amenable equivalence relation, then there exists $\varphi\in A$ such that $\chi_g^*\varphi=\varphi$ for all $g\in G$.*

Proof. Since G is countable, we can find a collection of $\lambda(G)$- and $\pi(G)$-invariant separable closed subspaces F_β of $l^\infty(G)$, containing the constant functions and such that $l^\infty(G)=\overline{\bigcup_{\beta\in B}F_\beta}$ (B is some index set).

For each finite sequence $\beta_1,\dots,\beta_n\in B$, set $F=F_{\beta_1}+\cdots+F_{\beta_n}$. As above, let $r: l^\infty(G)^*\to F^*$ denote the restriction map. Let $(A_{F,s})_{s\in S}$ be the Borel field of compact convex subsets of F_1^* of Lemma 2.7.

Set $I_F=\{\varphi\in E_1^*;\ r(\varphi)\in r(A)$ and $\chi_g^*r(\varphi)=r(\varphi),\ g\in G\}$.

Assertion 1. I_F is a non-empty $\sigma(E^*, E)$-closed set.

As r is continuous, A is compact and $\chi_g^*\in H(E_1^*)$, I_F is closed.

As $\mathcal{R}_G$ is an amenable equivalence relation, there exists a Borel function $\eta: S\to F_1^*$ with

$$\eta(s)\in A_{F,s}\quad\text{a.e.,}\tag{2.8.1}$$

$$\pi_{(s,t)}^*\eta(t)=\eta(s)\quad\text{a.e.}\tag{2.8.2}$$

Let η denote the element of $L^1(S, F)^*$ corresponding to $s\mapsto\eta(s)$. By (2.8.2) and the definition of χ^*, we have $\chi_g^*\eta=\eta$ for all $g\in G$ and by (2.8.1), $\eta\in\mathcal{M}_{L^1(S,F)}$. By the Hahn–Banach theorem, one can extend η to $\varphi\in F_1^*$ such that $r(\varphi)=\eta$. One can choose φ in $\mathcal{M}_E$ by the monotone extension theorem. Indeed, E is an ordered linear space and $L^1(S, F)$ is a cofinal subspace of E (see [**NB**, p 181]). Therefore, I_F is non-empty.

As E_1^* is $\sigma(E^*, E)$-compact, there exists $\varphi\in\bigcap_{\beta\in B}I_{F_\beta}$, i.e., φ is a mean of E such that for all $\beta\in B$, $r_\beta(\varphi)\in r_\beta(A)$ and $\chi_g^*r_\beta(\varphi)=r_\beta(\varphi)g\in G$ (where $r_\beta: E^*\to L^1(S, F_\beta)^*$).

To finish the proof of Lemma 2.8, we must prove the following

Assertion 2. $\varphi\in A$ and $\chi_g^*\varphi=\varphi$, for all $g\in G$.

If $\varphi\notin A$, then there exists $f\in E$ such that

$$\langle\varphi,f\rangle>\sup_{\psi\in A}\langle\psi,f\rangle.$$

Set $\delta = \langle\varphi, f\rangle - \sup_{\psi\in A}\langle\psi, f\rangle$. The simple functions of the type

$$\sum_{k=1}^{m} f_k\chi_{A_k},$$

where $f_k \in l^\infty(G)$ and $(A_k)_{1\le k\le n}$ is a measurable partition of S, are dense in E (see [**E**, (8.18.1)]). Therefore, there exist $\beta_1,\ldots,\beta_n \in B$, $F = F_{\beta_1}+\cdots+F_{\beta_n}$ and $\tilde f \in L^1(S, F)$ such that

$$\|f-\tilde f\| \le \tfrac12\delta.$$

Then as $\langle r_F(\varphi), \tilde f\rangle = \langle\varphi, \tilde f\rangle$ and $r_F(\varphi)\in r_F(A)$, we deduce that

$$\langle\varphi, f\rangle = \langle\varphi, \tilde f\rangle + \langle\varphi, f-\tilde f\rangle \le \sup_{\psi\in A}\langle\psi, \tilde f\rangle + \tfrac12\delta,$$

a contradiction.

We check the second part of the assertion in the same way. □

Let (S, μ) and G be as above. We may assume that S is a Polish space. Then the space $S\times\prod_{n\ge1} G$ is also Polish.

LEMMA 2.9. *$\tilde S = \{(s, (g_n)_{n\ge1});\ g_n\in G_s\}$ is a Borel subset of $S\times\prod_{n\ge1} G$.*

Proof. As $s\mapsto G_s$ is (Effros) Borel, then for fixed $g\in G$, $s\mapsto d(G_s, g)$ is Borel where d is the discrete metric ($d(g,h)=1$ if $g\ne h$) (see [**S1**, Theorem 2.5]). Therefore, for $\bar g = (g_n)_{n\ge1}\in\prod_{n\ge1} G$ fixed, the map $s\mapsto\sum_{n\ge1}(1/2^n)\,d(G_s, g_n)$ is also Borel. Moreover, for $s\in S$ fixed, $\bar g\in\prod_{n\ge1} G\mapsto\sum_{n\ge1}(1/2^n)\,d(G_s, g_n)$ is continuous. Hence the map

$$(s,\bar g)\in S\times\prod_{n\ge1} G\mapsto\sum_{n\ge1}\frac{1}{2^n}\,d(G_s, g_n)$$

is Borel and $\tilde S = \{(s,\bar g);\ \sum_{n\ge1}(1/2^n)\,d(G_s, g_n)=0\}$ is Borel. □

By [**Ch**, Theorem 2.6], $\tilde S$ endowed with the subspace Borel structure is a standard Borel space. Similarly, for each $n\ge1$,

$$\tilde S_n = \{(s, g_1^s,\ldots,g_n^s);\quad s\in S,\quad g_k^s\in G_s\quad\text{for}\quad 1\le k\le n\}$$

is a standard Borel space.

For each $s\in S$ and $n\ge1$, let us write $\Omega_s = \prod_{m\ge1} G_s$, $\Omega_{s,n} = \prod_{k=1}^n G_s$, and $\Omega_s^n = \prod_{k\ge n} G_s$.

If $(\eta_k)_{k\ge1}$ is a sequence of positive elements of $L^\infty(S, l^1(G))_1$, let us introduce the following notation:

$v_{s,n}$ is the product measure $\bigotimes_{k=1}^n \eta_k(s)$ on $\Omega_{s,n}$,

v_s(resp. v_s^n) is the infinite product measure $\bigotimes_{k\ge1}\eta_k(s)$(resp. $\bigotimes_{k\ge n}\eta_k(s)$) on Ω_s(resp. Ω_s^n),

$\tilde\mu$ is the measure $\int_S v_s\,d\mu(s)$ on $\tilde S$, and

$\tilde\mu_n$ is the measure $\int_S v_{s,n}\,d\mu(s)$ on $\tilde S_n$.

For each $g\in G$ and $\eta\in L^\infty(S, l^1(G))$, recall that χ_g is defined by

$$\chi_g\eta(s) = \pi_{\gamma(s,sg)}\eta(sg),\quad s\in S.$$

We shall write

$\chi_g v_{s,n}$ to mean the product measure $\bigotimes_{k=1}^n \chi_g \eta_k(s)$ on $\Omega_{s,n}$,

$\chi_g v_s$(resp. $\chi_g v_s^n$) to mean the infinite product measure $\bigotimes_{k=1} \chi_g \eta_k(s)$ on Ω_s(resp. $\bigotimes_{k\geq n} \chi_g \eta_k(s)$ on Ω_s^n).

We shall write $G = \{g_0 = e, g_1, g_2, \ldots\}$, and $G_k = \{g_0, g_1, \ldots, g_k\}$ for all $k \geq 0$. For each $g_k \in G$, denote by $\bar{g}_k : S \to G$ the map given by

$$\bar{g}_k = \begin{cases} e & \text{if } g_k \notin G_s, \\ g_k & \text{if } g_k \in G_s. \end{cases}$$

PROPOSITION 2.10. *Let (S, μ) and G be as above and assume that*

(i) *the induced equivalence relation $\mathcal{R}_G$ on S is amenable, and*

(ii) *the stabilizer G_s is amenable, μ-a.e.*

Then there exists a sequence $(\eta_n)_{n\geq 1}$ of elements of $P(S)$ such that

(a) *for $1 \leq k \leq n$, $\int_S \|\chi_{g_k}\eta_n(s) - \eta_n(s)\|_1 \, d\mu(s) \leq 1/2^n$, and*

(b) *if p_n is the smallest integer l such that for $0 \leq k \leq n-1$,*

$$\int_S d\mu(s)\chi_g v_{s,n-1}(\{(s, g_1^s, \ldots, g_{n-1}^s) \in \tilde{S}_{n-1};\, g_1^s \ldots g_{n-1}^s \in G_l\}) \geq 1 - n^{-1},$$

then for $g_k \in \bigcup_{h \in G_{p_n}} h^{-1} G_n h$,

$$\int_S \|\bar{g}_k \eta_n(s) - \eta_n(s)\|_1 \, d\mu(s) \leq \frac{1}{n(p_n+1)2^n}.$$

Proof. Let $\varphi \in \mathcal{M}_{E,c}$ be as provided by Lemma 2.8. As $P(S)$ is weak*-dense in $\mathcal{M}_{E,c}$, there exists a net $(v_\sigma)_{\sigma\in\Sigma}$ in $P(S) \subset L^1(S \times G)$ such that

$$\hat{v}_\sigma \to \varphi \text{ weak* in } E^*.$$

By Lemma 2.8, we have that for all $k \geq 1$,

$$(\chi_{g_k}^* \hat{v}_\sigma - \hat{v}_\sigma) \quad \text{and} \quad (\bar{g}_k^* \hat{v}_\sigma - \hat{v}_\sigma) \quad \text{converge weak* to 0 in } E^*.$$

Note that

$$L^1(S \times G)^* = L^\infty(S \times G) \subset L^1(S, l^\infty(G)) = E.$$

Therefore, for all $k \geq 1$,

$$(\chi_{g_k} v_\sigma - v_\sigma) \quad \text{and} \quad (\bar{g}_k v_\sigma - v_\sigma) \quad \text{converge weakly to 0 in } L^1(S \times G). \tag{2.10.1}$$

For each $k \geq 1$, consider the maps

$$\begin{aligned} N_k : L^1(S \times G) &\to L^1(S \times G) \times L^1(S \times G) \\ f \quad &\mapsto \quad (\chi_{g_k} f - f, \bar{g}_k f - f), \end{aligned}$$

and

$$\begin{aligned} N : L^1(S \times G) &\to \prod_{k\geq 1} (L^1(S \times G) \times L^1(S \times G)) \\ f \quad &\mapsto \quad (\chi_{g_k} f - f, \bar{g}_k f - f)_{k\geq 1}. \end{aligned}$$

Recall (see [**B**, II.6.6, Proposition 8]) that the

$$\sigma\left(\prod_{k\geq 1}(L^1(S\times G)\times L^1(S\times G)),\bigotimes_{k\geq 1}(L^\infty(S\times G)\times L^\infty(S\times G))\right)\text{-topology}$$

on $\prod_{k\geq 1}(L^1(S\times G)\times L^1(S\times G))$ is the product of the $\sigma(L^1(S\times G), L^\infty(S\times G))$-topologies on the various copies of $L^1(S\times G)$.

Set $B=\text{Conv}\{\hat{v}_\sigma; \sigma\in\Sigma\}$. By (2.10.1), we have $0\in\overline{N(B)}^{\text{weak}}$. As $N(B)$ is convex, 0 belongs to the norm closure of $N(B)$. Hence there exists a net $(\eta_i)_{i\in I}$ such that for all $k\geq 1$,

$$\|\chi_{g_k}\eta_i-\eta_i\|\to 0 \quad\text{and}\quad \|\bar{g}_k\eta_i-\eta_i\|\to 0 \quad (\text{in } L^1(S\times G)).$$

The proposition follows easily by induction. □

Let $\tilde{S}=\{(s,\bar{g}^s); s\in S, \bar{g}^s=(g_n^s)_{n\geq 1}\in\Omega_s\}$ be as in Lemma 2.9 and let $(\eta_n)_{n\geq 1}$ be a *sequence in* $P(S)$ *as in Proposition 2.10.*

Keeping the notation introduced after Lemma 2.9, let us consider the probability measure

$$\tilde{\mu}=\int_S v_s\, d\mu(s),$$

where v_s is the infinite product measure $\bigotimes_{n\geq 1}\eta_n(s)$ on Ω_s, $s\in S$.

Consider the measured equivalence relation $\tilde{\mathscr{R}}$ on $(\tilde{S},\tilde{\mu})$ defined for $\tilde{s}=(s,\bar{g}^s)$ and $\tilde{t}=(t,\bar{g}^t)\in\tilde{S}$ by

$$\tilde{s}\tilde{\mathscr{R}}\tilde{t}\Leftrightarrow s\mathscr{R}_G t \quad\text{and for some } m<\infty,\quad \gamma(t,s)g_n^s\gamma(s,t)=g_n^t \quad\text{for } n\geq m+1.$$

As G is countable, each $\tilde{\mathscr{R}}$-equivalence class is countable, i.e., $\tilde{\mathscr{R}}$ is discrete.

Let us denote by $\tilde{\mathscr{R}}_{\gamma,G}\subset\tilde{\mathscr{R}}$ the sub equivalence relation given by

$$\tilde{s}\tilde{\mathscr{R}}_{\gamma,G}\tilde{t}\Leftrightarrow s\mathscr{R}_G t \quad\text{and}\quad \gamma(t,s)g_n^s\gamma(s,t)=g_n^t \quad\text{for } n\geq 1.$$

LEMMA 2.11. *$\tilde{\mathscr{R}}_{\gamma,G}$ is non singular.*

Proof. Let us first check that, for each $g\in G$ and almost every $s\in S$, the product measures v_s and $\chi_g v_s$ on Ω_s are equivalent. For each $n\geq 1$, set

$$\varphi_n(s)(g_n^s)=\frac{\chi_g\eta_n(s)}{\eta_n(s)}(g_n^s)=\frac{\eta_n(sg)(\gamma(sg,s)g_n^s\gamma(s,sg))}{\eta_n(s)(g_n^s)}.$$

For each $n\geq 1$, $\eta_n(s)$ and $\chi_g\eta_n(s)$ are equivalent (as $\eta_n\in P(S)$), and so by Kakutani's criterion (see [**HS**, 22.36]), what we must show is that

$$\prod_{n\geq 1}\|\varphi_n^{1/2}(s)\|_{1,\eta_n(s)}>0. \tag{2.11.1}$$

As $0<\|\varphi_n^{1/2}(s)\|_{1,\eta_n(s)}\leq\|\varphi_n(s)\|_{1,\eta_n(s)}^{1/2}=1$, it is equivalent to show that

$$\sum_{n\geq 1}(1-\|\varphi_n^{1/2}(s)\|_{1,\eta_n(s)})\leq\infty.$$

On the one hand, we have

$$\sum_{n\geq 1}(1-\|\varphi_n^{1/2}(s)\|_{1,\eta_n(s)})\leq\sum_{n\geq 1}\|1-\varphi_n(s)\|_{1,\eta_n(s)},$$

and, on the other hand, by Proposition 2.10(a) we have

$$\sum_{n\geq 1} \|1-\varphi_n(s)\|_{1,\eta_n(s)} = \sum_{n\geq 1} \|\chi_g\eta_n(s)-\eta_n(s)\|_1 < \infty.$$

Denote by $\tilde{r}:\tilde{\mathcal{R}}_{\gamma,G}\to\mathbb{R}_+$ the Radon–Nikodym cocycle. As for $\tilde{s}=(s,\bar{g}^s)\in\tilde{S}$,

$$\tilde{r}(\tilde{s},\tilde{s}g)=r(s,sg)\frac{d\chi_g v_s}{dv_s}(\bar{g}^s),$$

the lemma is proved. □

Consider the Borel homomorphism $\delta:\tilde{\mathcal{R}}\to\mathbb{R}_+$ defined for $(\tilde{s},\tilde{t})\in\tilde{\mathcal{R}}$ by

$$\sigma(\tilde{s},\tilde{t})=r(s,t)\prod_{n\geq 1}\frac{\eta_n(t)(g_n^t)}{\eta_n(s)(g_n^s)}.$$

This is the Radon–Nikodym cocycle of $\tilde{\mu}$. With the notation of Lemma 2.11, for $(\tilde{s},\tilde{t})\in\tilde{\mathcal{R}}$, there exists $m<\infty$ such that

$$\sigma(\tilde{s},\tilde{t})=\tilde{r}(\tilde{s},\tilde{t})\prod_{n=1}^{m}\frac{\eta_n(t)(g_n^t)}{\eta_n(t)(\gamma(t,s)g_n^s\gamma(s,t))}.$$

Therefore, for $(\tilde{s},\tilde{t})\in\tilde{\mathcal{R}}$, $\delta(\tilde{s},\tilde{t})\in\,]0,\infty[$. This shows that $\tilde{\mathcal{R}}$ is non-singular.

Let us show now that $\tilde{\mathcal{R}}$ is amenable. To do so, we will use results of [**FSZ**] and [**S2**].

Let $\mathcal{R}_T$ denote the equivalence relation on $(\tilde{S},\tilde{\mu})$ given by

$$\tilde{s}\mathcal{R}_T\tilde{t}\Leftrightarrow s=t \quad\text{and there exists } m<\infty \text{ such that}\quad g_n^s=g_n^t \quad\text{for } n\geq m+1.$$

As $\gamma(s,s)=1\in G$ for all $s\in S$, $\mathcal{R}_T\subset\tilde{\mathcal{R}}$.

LEMMA 2.12. *$\mathcal{R}_T\subseteq\tilde{\mathcal{R}}$ is normal (in the sense of* [***FSZ***, § 2, *Theorem* 2.2]).

Proof. As above denote by $\mathcal{R}_G$ the equivalence relation generated by the action of G on S, and let $\Theta:\tilde{\mathcal{R}}\to\mathcal{R}_G$ denote the homomorphism defined by $\Theta(\tilde{s},\tilde{t})=(s,sg)$ where $sg=t$. If $(\tilde{s},\tilde{t})\in\tilde{\mathcal{R}}$, then

$$\theta(\tilde{s},\tilde{t})\in\mathcal{R}_G^{(0)}=S \quad\text{if and only if}\quad (\tilde{s},\tilde{f})\in\mathcal{R}_T.$$

Hence, ker $\Theta=\mathcal{R}_T$.

Moreover, let $(sg^{-1},s)\in\mathcal{R}_G$ and $\tilde{s}=(s,\bar{g}^s)\in\tilde{S}$, so that $\Theta(\tilde{s})=s$, the source of (sg^{-1},s). Set $\tilde{t}=(sg^{-1},\gamma(sg^{-1},s)\bar{g}^s\gamma(sg^{-1},s))\in\tilde{S}$. Then $(\tilde{t},\tilde{s})\in\tilde{\mathcal{R}}$ and $\Theta(\tilde{t},\tilde{s})=(sg^{-1},s)$.

Therefore, by [**FSZ**], Theorem 2.2(d), to finish the proof of Lemma 2.12, we have only to show that for any discrete ergodic measured groupoid $(\mathcal{S}',v')$ and homomorphism $\Theta':\tilde{\mathcal{R}}\to\mathcal{S}'$ with ker $\Theta'\supset\mathcal{R}_T$, there is a homomorphism $\kappa:\mathcal{R}_G\to\mathcal{S}'$ with $\kappa\Theta=\Theta'$.

Denote by $p:\tilde{S}\to S$ the canonical projection. For each $s\in S$, the relation $\mathcal{R}_T$ is ergodic on $(p^{-1}(s),v_s)$. The preceding property can now be shown as in [**FSZ**, Theorem 2.2 (p. 249)]. □

By [**S2**, Theorem 3.7], as $\mathcal{R}_G$ is amenable, $\mathcal{R}_T\subseteq\tilde{\mathcal{R}}$ is relatively amenable.

For each $s\in S$, let $\mathcal{R}_{T,s}$ denote the discrete measured ergodic equivalence relation on (Ω_s,v_s) consisting of tail equivalence.

Then $\mathcal{R}_{T,s}$ is amenable for each $s\in S$. As $W^*(\mathcal{R}_T)\cong\int_S^{\oplus}W^*(\mathcal{R}_{T,s})\,d\mu(s)$, it follows that the equivalence relation $\mathcal{R}_T$ is amenable.

Therefore, by [**S2**, Corollary 3.4] we have

LEMMA 2.13. $\tilde{\mathcal{R}}$ *is amenable.*

If $\tilde{s}\tilde{\mathcal{R}}\tilde{t}$, there exists $m<\infty$ such that $(s,t)\in\mathcal{R}_G$ and $\gamma(t,s)g_n^s\gamma(s,t)=g_n^t$ for $n\geq m+1$. Consider the map $\alpha:\tilde{\mathcal{R}}\to G$ defined by

$$\alpha((\tilde{s},\tilde{t}))=g_1^s\dots g_m^s\gamma(s,t)(g_m^t)^{-1}\dots(g_1^t)^{-1}.$$

As γ is a homomorphism, α is also a homomorphism.

Let m be a probability measure on G equivalent to the counting measure. On $(\tilde{S}\times G,\tilde{\mu}\times m)$, we still dentoe by $\tilde{\mathcal{R}}$ the equivalence relation given by

$$(\tilde{s},g)\tilde{\mathcal{R}}(\tilde{t},h)\Leftrightarrow\tilde{s}\tilde{\mathcal{R}}\tilde{t}\quad\text{and}\quad g\alpha(\tilde{s},\tilde{t})=h.$$

On $(S\times G,\mu\times m)$, let $\tilde{\mathcal{R}}_G$ denote the equivalence relation defined by

$$(s,g)\tilde{\mathcal{R}}_G(t,h)\Leftrightarrow s\mathcal{R}_Gt\quad\text{and}\quad g\gamma(s,t)=h.$$

Consider the action β of G on $(S\times G,\mu\times m)$ defined by

$$(s,h)\beta(g)=(sg,hg)\quad\text{for all}\quad(s,h)\in S\times G.$$

Remark 2.14. For every $\bar{g}\in\mathcal{G}=\{\bar{g}:s\mapsto g_s;\ \bar{g}\text{ a Borel map with }\bar{g}(s)\in G_s\}$ and every $f\in L^\infty(S\times G)$, we denote by $\rho(\bar{g})f$ the function

$$\rho(\bar{g})f(s,h)=f(s,hg_s).$$

If $f\in L^\infty(S\times G)$ is $\tilde{\mathcal{R}}_G$- and ρ-invariant, then f is β-invariant.

Let $\Pi:L^\infty(\tilde{S}\times G,\tilde{\mu}\times m)\to L^\infty(S\times G,\mu\times m)$ denote the map defined by

$$\Pi f(s,k)=\int_{\Omega_s}f(s,\bar{g}^s,k)\,dv_s(\bar{g}^s).$$

Π has the following properties:

(1) Π is a norm one linear projection, and hence a conditional expectation on considering $L^\infty(S\times G,\mu\times m)$ as a subalgebra of $L^\infty(\tilde{S}\times G,\tilde{\mu}\times m)$.

(2) Π intertwines the left regular representations (for $f\in L^\infty(\tilde{S}\times G,\tilde{\mu}\times m)$, $g\in G$, one has $\lambda(g)f(\tilde{s},h)=f(\tilde{s},g^{-1}h)$).

LEMMA 2.15. $\Pi(L^\infty(\tilde{S}\times G,\tilde{\mu}\times m)^{\tilde{\mathcal{R}}})\subset L^\infty(S\times G,\mu\times m)^\beta$.

Proof. Let $f\in L^\infty(\tilde{S}\times G,\tilde{\mu}\times m)^{\tilde{\mathcal{R}}}$, $\|f\|_\infty\leq 1$. By Remark 2.14, it suffices to prove that

(i) for all $\bar{g}\in\mathcal{G}$, $\rho(\bar{g})\Pi f=\Pi f$,

(ii) Πf is $\tilde{\mathcal{R}}_G$-invariant.

As m is a probability measure, (i) and (ii) will follow if we show that for all $h\in G$, and all $\varepsilon>0$,

(i′) for all $\bar{g}\in\mathcal{G}$, $\|\Pi f(\cdot,h)-\rho(\bar{g})\Pi f(\cdot,h)\|_1\leq\varepsilon$,

(ii′) for all $g\in G$, $\|\Pi f(\cdot,h)-\chi_g\Pi f(\cdot,h)\|_1\leq\varepsilon$.

Let us verify (i′). Let $h\in G$, $\bar{g}\in\mathcal{G}$ and $\varepsilon>0$ be given. Let $n_0\geq 0$ be such that if $B=\coprod_{l=0}^{n_0}\{s\in S;\ \bar{g}(s)=g_l\}$, then

$$\mu(B)\geq 1-\tfrac{1}{4}\varepsilon.$$

Let us fix $n\geq n_0$, with $2^{-(n-1)}\leq\varepsilon$. With p_n as provided by Proposition 2.10, set

$$A_{s,n}=\{(g_1^s,\dots,g_{n-1}^s)\in\Omega_{s,n-1};\ g_1^s\dots g_{n-1}^s\in G_{p_n}\}.$$

To simplify notation, let us set

$$\bar{g}_n^s=(g_k^s)_{k\geq n}\in\Omega_{s,n},\quad \tilde{g}_{n-1}^s=(g_1^s,\dots,g_{n-1}^s),$$

and

$$k(s)=(g_1^s\dots g_{n-1}^s)^{-1}\bar{g}(s)(g_1^s\dots g_{n-1}^s).$$

Note that

$$(s,\bar{g}^s,h\bar{g}(s))\tilde{\mathscr{R}}(s,\tilde{g}_{n-1}^s,k(s)g_n^s,\bar{g}_{n+1}^s,h).$$

Then, as f is $\tilde{\mathscr{R}}$-invariant, we have

$$\begin{aligned}
&\|\Pi f(\cdot,h)-\rho(\bar{g})\Pi f(\cdot,h)\|_1\\
&=\int_S d\mu(s)\left|\int_{\Omega_s}[f(s,\bar{g}^s,h)-f(s,\bar{g}^s,h\bar{g}_k(s))]\,d\nu_s(\bar{g}^s)\right|\\
&=\int_S d\mu(s)\left|\int_{\Omega_s}[f(s,\bar{g}^s,h)-f(s,\tilde{g}_{n-1}^s,k(s)g_n^s,\bar{g}_{n+1}^s,h)]\,d\nu_s(\bar{g}^s)\right|\\
&=\int_S d\mu(s)\left|\int_{\Omega_s}f(s,\bar{g}^s,h)[d\nu_s(\bar{g}^s)-d\nu_{s,n-1}(\tilde{g}_{n-1}^s)\,d\eta_n(s)\right.\\
&\qquad\left.\times(k(s)^{-1}g_n^s)\,d\nu_{s,n+1}(\bar{g}_{n+1}^s)]\right|\\
&\leq\int_S d\mu(s)\int_{\Omega_{s,n-1}}d\nu_{s,n-1}(\tilde{g}_{n-1}^s)\|\eta_n(s)-k(s)\eta_n(s)\|_1\\
&\leq\tfrac14\varepsilon+\int_B d\mu(s)\int_{A_{s,n-1}}d\nu_{s,n-1}(\tilde{g}_{n-1}^s)\|\eta_n(s)-k(s)\eta_n(s)\|_1.
\end{aligned}$$

Set $G_{n,p_n}=\bigcup_{p\in p_n}p^{-1}G_np$. Then for all $s\in B$ and $\tilde{g}_{n-1}^s\in A_{s,n-1}$, we have $k(s)\in G_{n,p_n}$. As $|G_{n,p_n}|\leq(p_n+1)n$, we obtain by Proposition 2.10,

$$\begin{aligned}
\|\Pi f(\cdot,h)-\rho(\bar{g})\Pi f(\cdot,h)\|_1&\leq\tfrac14\varepsilon+\sum_{g_k\in G_{n,p_n}}\int_B d\mu(s)\|\eta_n(s)-\bar{g}_k\eta_n(s)\|_1\\
&\leq\tfrac14\varepsilon+(p_n+1)n\frac{1}{(p_n+1)n2^n}\leq\tfrac12\varepsilon+\tfrac12\varepsilon=\varepsilon.
\end{aligned}$$

Let us prove (ii′). Let $g,h\in G$ and $\varepsilon>0$ be given. For each $l\geq 1$, let us denote by $\bar{e}_l=(e,\dots,e)$ the neutral element of $\prod_{n=1}^{l}G$. Let us choose n such that $g\in G_n$ and $n\geq 6/\varepsilon$.

To simplify notation, set $t=sg$. Then $(s,\bar{g}^s,h)$ and $(t,\gamma(t,s)\bar{g}^s\gamma(s,t),h\gamma(s,t))$ are $\tilde{\mathscr{R}}$-equivalent. Therefore, we have

$$\begin{aligned}
&\|\Pi f(\cdot,h)-\chi_g\Pi f(\cdot,h)\|_1\\
&=\int_S d\mu(s)\left|\int_{\Omega_s}f(s,\bar{g}^s,h)\,d\nu_s(\bar{g}^s)-\int_{\Omega_t}f(t,\bar{g}^t,h\gamma(s,t))\,d\nu_t(\bar{g}^t)\right|\\
&=\int_S d\mu(s)\left|\int_{\Omega_s}f(s,\bar{g}^s,h)[d\nu_s(\bar{g}^s)-d\nu_t^n(\gamma(t,s)\bar{g}_n^s\gamma(s,t))]\right|\\
&=\int_S d\mu(s)\left|\int_{\Omega_s}f(s,\bar{g}^s,h)[d\nu_s(\bar{g}^s)-d\chi_g\nu_s^n(\bar{g}_n^s)]\right|.
\end{aligned}$$

Moreover, for all $\bar{g}^s \in \Omega_s$,

$$f(s, \bar{g}^s, h) = f(s, \bar{e}_{n-1}, (g_1^s, \ldots, g_{n-1}^s) g_n^s, \bar{g}_{n+1}^s, h).$$

Therefore, by Proposition 2.10, we have (2.15.3)

$$\int_S d\mu(s) \left| \int_{\Omega_s} f(s, \bar{g}^s, h)\, dv_s(\bar{g}^s) - \int_{\Omega_s^n} f(s, \bar{e}_{n-1}, \bar{g}_n^s, h)\, dv_s^n(\bar{g}_n^s) \right|$$

$$= \int_S d\mu(s) \left| \int_{\Omega_{s,n-1}} dv_{s,n-1}(\tilde{g}_{n-1}^s) \int_{\Omega_s^n} f(s, \bar{e}_{n-1}, \bar{g}_n^s, h) \right.$$

$$\left. \times [d\eta_n(s)((g_1^s \ldots g_{n-1}^s)^{-1} g_n^s) - d\eta_n(s)(g_n^s)]\, dv_s^{n+1}(\bar{g}_{n+1}^s) \right|$$

$$\leq \int_S d\mu(s) \int_{\Omega_{s,n-1}} dv_{s,n-1}(\tilde{g}_{n-1}^s) \|(g_1^s \ldots g_{n-1}^s)\eta_n(s) - \eta_n(s)\|_1$$

$$\leq \frac{2}{n} + \int_S d\mu(s) \int_{A_{s,n-1}} dv_{s,n-1}(\tilde{g}_{n-1}^s) \|(g_1^s \ldots g_{n-1}^s)\eta_n(s) - \eta_n(s)\|_1$$

$$\leq \frac{2}{n} + \sum_{l=1}^{p_n} \int_S d\mu(s) \|\bar{g}_l^s \eta_n(s) - \eta_n(s)\|_1$$

$$\leq \frac{1}{n}[2 + 2^{-n}].$$

Similarly we have (2.15.4)

$$\int_S d\mu(s) \left| \int_{\Omega_s} f(s, \bar{g}^s, h)\, d\chi_g v_s(\bar{g}^s) - \int_{\Omega_s^n} f(s, \bar{e}_{n-1}, \bar{g}_n^s, h)\, d\chi_g v_s^n(\bar{g}_n^s) \right|$$

$$\leq \int_S d\mu(s) \int_{\Omega_{s,n-1}} dv_{s,n-1}(\tilde{g}_{n-1}^s) \|(g_1^s \ldots g_{n-1}^s)\chi_g \eta_n(s) - \chi_g \eta_n(s)\|_1$$

$$\leq \int_S d\mu(s) \int_{\Omega_{s,n-1}} dv_{s,n-1}(\tilde{g}_{n-1}^s)[\|(g_1^s \ldots g_{n-1}^s)(\chi_g \eta_n(s) - \eta_n(s))\|_1$$

$$+ \|(g_1^s \ldots g_{n-1}^s)\eta_n(s) - \eta_n(s)\|_1 + \|\chi_g \eta_n(s) - \eta_n(s))\|_1]$$

$$\leq 2 \int_S d\mu(s) \|\chi_g \eta_n(s) - \eta_n(s))\|_1 + \frac{2}{n} + \sum_{l=1}^{p_n} \int_S d\mu(s) \|\bar{g}_l^s \eta_n(s) - \eta_n(s)\|_1$$

$$\leq \frac{1}{n}[2 + 2^{-n}] + \frac{1}{2^{n-1}}.$$

Hence by (2.15.2), (2.15.3) and (2.15.4), we have

$$\|\Pi f(\cdot, h) - \chi_g \Pi f(\cdot, h)\|_1$$

$$\leq \frac{6}{n} + \int_S d\mu(s) \left| \int_{\Omega_s^n} f(s, \bar{e}_{n-1}, \bar{g}_n^s, h)[dv_s^n(\bar{g}_n^s) - d\chi_g v_{sg}^n(\bar{g}_n^s)] \right|$$

$$\leq \frac{6}{n} + \int_S d\mu(s) \int_{\Omega_s^n} |dv_s^n(\bar{g}_n^s) - d\chi_g v_s^n(g_n^s)|.$$

As in Lemma 2.11, for $n \geq 1$, set

$$\varphi_n(s)(g_n^s) = \frac{\eta_n(sg)(\gamma(sg, g)g_n^s\gamma(s, sg))}{\eta_n(s)(g_n^s)}.$$

We have

$$\int_{\Omega_s^n} |d\nu_s^n(\bar{g}_n^s) - d\chi_g \nu_s^n(\bar{g}_n^s)| = \int_{\Omega_s^n} \left|1 - \prod_{k \geq n} \varphi_k(s)(g_k(s))\right| d\nu_s^n(\bar{g}_n^s)$$

$$\leq \sum_{k \geq n} \int_{G_s} |1 - \varphi_k(s)(g_k^s)| \, d\eta_k(s)(g_k^s)$$

$$\leq \sum_{k \geq n} \|\eta_k(s) - \chi_g \eta_k(s)\|_1.$$

Hence,

$$\|\Pi f(\cdot, h) - \chi_g \Pi f(\cdot, h)\|_1 \leq \frac{5}{n} + \frac{1}{2^{n-1}} + \int_S d\mu(s) \sum_{k \geq n} \|\eta_k s - \chi_g \eta_k(s)\|_1$$

$$\leq \frac{5}{n} + \frac{1}{2^{n-1}} \leq \varepsilon. \qquad \square$$

LEMMA 2.16. *The map* $\Pi: L^\infty(\tilde{S} \times G, \tilde{\mu} \times m)^{\tilde{\mathscr{R}}} \to L^\infty(S \times G, \mu \times m)^\beta$ *is bijective.*

Proof. By considering $L^\infty(S \times G, \mu \times m)$ as a subalgebra of $L^\infty(\tilde{S} \times G, \tilde{\mu} \times m)$ and recalling the definition of $\tilde{\mathscr{R}}$, we see that Π is surjective.

Let $f_1, f_2 \in L^\infty(\tilde{S} \times G, \tilde{\mu} \times m)_1^{\tilde{\mathscr{R}}}$ with $\Pi f_1 = \Pi f_2$. For all $h \in G$ and $\varepsilon > 0$, let us show that

$$\int_{\tilde{S}} |f_1(s, \bar{g}^s, h) - f_2(s, \bar{g}^s, h)| \, d\tilde{\mu}(s, \bar{g}^s) \leq 12\varepsilon.$$

For each $n \geq 1$ and each $i = 1, 2$, set $E_n f_i(s, \bar{g}^s, h) = \int_{\Omega_s^n} f_i(s, \bar{g}^s, h) \, d\nu_s^n(\bar{g}_n^s)$. Then there exists N_0 such that for $n \geq N_0$,

$$\int_{\tilde{S}} |f_i(s, \bar{g}^s, h) - E_n f_i(s, \bar{g}^s, h)| \, d\tilde{\mu}(s, \bar{g}^s) < \tfrac{1}{4}\varepsilon. \tag{2.16.1}$$

Let $n \geq N_0$ be such that $1/n \leq \varepsilon/24$. Then by (2.16.1), we have for $i = 1, 2$,

$$\int_S \left|\Pi f_i(s, h) - \int_{\Omega_s^n} f_i(s, \bar{e}_{n-1}, \bar{g}_n^s, h) \, d\nu_s^n(\bar{g}_n^s)\right| d\mu(s)$$

$$= \int_S |\Pi f_i(s, h) - E_n f_i(s, \bar{e}_{n-1}, h)| \, d\mu(s) \leq \frac{3}{n}.$$

Similarly, we have

$$\int_S d\mu(s) \int_{\Omega_{n-1,s}} |E_n f_i(s, \tilde{g}_{n-1}^s, h) - E_n f_i(s, \bar{e}_{n-1}, h)| \, d\nu_{n-1,s}(\tilde{g}_{n-1}^s) \leq \frac{3}{n}.$$

Therefore,

$$\int_{\tilde S} |f_1(s, \bar g_1^s h) - f_2(s, \bar g^s, h)|\, d\tilde\mu(\tilde s)$$

$$\le \tfrac{1}{2}\varepsilon + \int_S d\mu(s) \int_{\Omega_{n-1,s}} |E_n f_1(s, \tilde g^s_{n-1}, h) - E_n f_2(s, \tilde g^s_{n-1}, h)|\, dv_{n-1,s}(\tilde g^s_{n-1})$$

$$\le \tfrac{1}{2}\varepsilon + \sum_{i=1}^{2} \int_S d\mu(s) \int_{\Omega_{n-1,s}} |E_n f_i(s, \tilde g^s_{n-1}, h) - E_n f_i(s, \bar e_{n-1}, h)|\, dv_{n-1,s}(\tilde g^s_{n-1})$$

$$+ \sum_{i=1}^{2} \int_S |\Pi f_i(s, h) - E_n f_i(s, \bar e_{n-1}, h)|\, d\mu(s)$$

$$\le \tfrac{1}{2}\varepsilon + \frac{12}{n} \le \varepsilon. \qquad \square$$

Proposition 2.17. *Let* $(\tilde S, \tilde\mu)$, $\tilde{\mathcal{R}}$, *and* α *be defined as after Proposition* 2.10 *and Lemma* 2.13.

It follows that the Mackey range of α *is isomorphic (as a G-space) to* (S, μ).

Proof. By Lemmas 2.15 and 2.16, the map

$$\Pi : L^\infty(\tilde S \times G, \tilde\mu \times m) \to L^\infty(S \times G, \mu \times m)$$

induces an isomorphism from $L^\infty(\tilde S \times G, \tilde\mu \times m)^{\tilde{\mathcal{R}}}$ to $L^\infty(S \times G, \mu \times m)^\beta$ which intertwines the left regular representations.

Let $p: S \times G \to S$ denote the Borel map $(s, h) \mapsto sh^{-1}$. Then for all $g \in G$,

$$p((s, h)\beta(g)) = p(sg, hg) = p(s, h),$$

$$p(\lambda(g)(s, h)) = p(s, g^{-1}h) = p(s, h)g.$$

Therefore, p induces a map from $L^\infty(S, \mu)$ to $L^\infty(S \times G, \mu \times m)^\beta$ which intertwines the left regular representation on $S \times G$ and the G-action on S.

If $f \in L^\infty(S \times G, \mu \times m)^\beta$, set $\bar f(s) = f(s, e)$, for $s \in S$.
Then $\bar f \in L^\infty(S, \mu)$ and $\bar f p = f$. Hence, $p: L^\infty(S, \mu) \to L^\infty(S \times G, \mu \times m)^\beta$ is an (isometric) isomorphism.

Therefore, the proposition is proved. $\square$

3. *Mackey ranges and Poisson boundaries*

Let G be a second countable locally compact group. Following [**CW**], let us recall the definition of a G-invariant matrix-valued random walk.

Let $(l_n)_{n \ge 0}$ be a sequence of integers ≥ 1 and denote by E_n the disjoint union of l_n copies of G. For each $n \ge 1$, let σ_n be a $(l_{n-1} \times l_n)$-matrix of positive measures on G such that

$$\sum_{j=1}^{l_n} \sigma_{n,i,j}(G) = 1 \quad \text{for} \quad 1 \le i \le l_{n-1}. \tag{3.1}$$

Such a matrix defines a transition probability P_n^{n-1} from E_{n-1} to E_n by

$$P_n^{n-1}((i, x), (j, A)) = \sigma_{n,i,j}(x^{-1}A),$$

where the point x is in the ith copy of G at the $(n-1)$st step and A is a Borel set in the jth copy of G at the nth step. Notice that for each $m\geq 0$, E_m is a G-space (by left translation of G on G) and that for $n\geq 1$,

$$P_n^{n-1}(g(i,x),g(j,A))=P_n^{n-1}((i,gx),(j,gA))=P_n^{n-1}((i,x),(j,A)).$$

By the right group-invariant matrix-valued random walk on G associated with the sequence $(\sigma_n)_{n\geq 1}$ is meant the Markov process consisting of the sequence of measurable spaces $(E_n)_{n\geq 0}$ with the transition probabilities $(P_n^{n-1})_{n\geq 1}$.

For each $n\geq 0$, let λ_n denote the measure on E_n extending Haar measure on each copy of G. We define a (bounded) harmonic function h for the matrix-valued random walk to be a sequence $h_n\in L^\infty(E_n,\lambda_n)$ such that

$$h_{n-1}((j,g))=\int_{E_n} h_n((k,x))P_n^{n-1}((j,g)\,d(k,x)) \quad \text{for} \quad (j,g)\in E_{n-1},$$

$$\sup_n \|h_n\|_\infty<\infty.$$

The space $\mathscr{H}^\infty$ of bounded harmonic functions is a Banach G-space with the norm $\|h\|=\sup_n \|h_n\|_\infty$ and the action of G induced by the left multiplication of G on E_n, $n\geq 0$.

Let σ_0 be a probability measure on E_0 equivalent to the measure λ_0. Let P denote the Markov measure on $(\Omega,\mathscr{A})=\prod_{n\geq 0}(E_n,\mathscr{A}_n)$ determined by the transition probabilities P_n^{n-1} and the initial distribution σ_0 (see for example [**N**, Proposition 5.2.1]). Let $\mathscr{A}_\infty\subset\mathscr{A}$ denote the asymptotic σ-algebra (or tail σ-algebra) of the matrix-valued random walk on G. Note that $\mathscr{A}_\infty$ is G-invariant with respect to the action of G on $(\Omega,\mathscr{A})$ given by

$$((\omega_n)_{n\geq 0})g=(g^{-1}\omega_n)_{n\geq 0} \quad \text{for} \quad (\omega_n)_{n\geq 0}\in\Omega. \tag{3.2}$$

The Poisson boundary of the matrix-valued random walk is a standard measured space (B,μ) such that

$$L^\infty(B,\mu)\cong L^\infty(\Omega,\mathscr{A}_\infty,P).$$

The action of G on $L^\infty(\Omega,\mathscr{A}_\infty,P)$ defined via (3.2) gives rise to a G-action on (B,μ).

For any $f\in L^\infty(\Omega,\mathscr{A}_\infty,P)$ and $n\geq 0$, define

$$h_n(f)(k,x)=\int_\Omega f(\omega)P_{n,(k,x)}(d\omega), \quad (k,x)\in E_n, \tag{3.3}$$

where $P_{n,(k,x)}$ is the Markov probability measure on $(\Omega,\mathscr{A}_\infty^n)$ defined by the transition probabilities $(P_n^{n-1})_{n\geq 1}$ (see, for example, [**N**, Proposition 5.2.1]). The formula (3.3) induces an isometric G-isomorphism between $L^\infty(\Omega,\mathscr{A}_\infty,P)$ and $\mathscr{H}^\infty$ (see [**N**, Proposition 5.2.2] and see [**J**] for a detailed proof).

From now on we shall assume that G is discrete. The main result of this section is the following:

THEOREM 3.1. *Let $(\tilde S,\tilde\mu)$ be a standard Borel space and $\tilde{\mathscr{R}}$ an amenable, ergodic, discrete equivalence relation on $(\tilde S,\tilde\mu)$. Let G be a (countable) discrete group and let $\alpha:\tilde{\mathscr{R}}\to G$ be a homomorphism. It follows that the Mackey range of α is isomorphic as a G-space to the Poisson boundary of a matrix-valued random walk on G.*

By the isomorphism between harmonic functions and L^∞-functions on the Poisson boundary, Theorem 3.1 follows immediately from

PROPOSITION 3.2. *Let* $(\tilde{S}, \tilde{\mu})$, $\tilde{\mathscr{R}}$, *and* α *be as in Theorem* 3.1. *There exists a sequence* $(\sigma_n)_{n\geq 1}$ *of* $(l_{n-1}\times l_n)$*-matrices of positive measures on* G *verifying* (3.1), *with* $l_0=1$, *such that the space of (bounded) harmonic functions of the right group-invariant matrix-valued random walk defined by* $(\sigma_n)_{n\geq 1}$ *is G-isomorphic to the Mackey range of* α.

The proof of Proposition 3.2 divides into two parts.

First part of the proof. Let X denote the compact group $\prod_{n\geq 1}\{0,1\}$ with the Borel σ-algebra $\mathscr{A}$ and denote by K the dense subgroup $\bigoplus_{n\geq 1}\mathbb{Z}/2\mathbb{Z}$ acting on X by addition. For all $k\geq 1$, let us denote by S_k the automorphism of X corresponding to the element $(x_n)_{n\geq 1}\in K$ with $x_n=\delta_{n,k}$, $n\geq 1$.

The equivalence relation $\tilde{\mathscr{R}}$ being amenable, by Theorem 10 of [**CFW**] there exist a probability measure on μ on $(X, \mathscr{A})$, which is non-atomic, quasi-invariant and ergodic under K, saturated null sets $\tilde{S}_0\subseteq\tilde{S}$ and $X_0\subseteq X$, and an isomorphism $\psi:\tilde{S}\backslash\tilde{S}_0\to X\backslash X_0$ such that

(1) $$\psi(\tilde{\mu}) \quad \text{is equivalent to } \mu,$$

(2) $$\psi(\{t\in\tilde{S};\, t\tilde{\mathscr{R}}s\})=\{x\in X;\, x\mathscr{R}_K\psi(s)\}, \quad s, t\in\tilde{S}\backslash\tilde{S}_0.$$

By [**C2**, Lemma 7], we may assume that

(3) for each $k\geq 1$, $\log d\mu S_k/d\mu$ takes only finitely many values.

Consider the homomorphism $\psi(\alpha):\mathscr{R}_K\to G$ defined (up to equivalence) by

$$\psi(\alpha)(\psi(s),\psi(t))=\alpha(s,t), \quad (s,t)\in\tilde{\mathscr{R}} \quad \text{and} \quad s,t\in\tilde{S}\backslash\tilde{S}_0,$$
$$\psi(\alpha)(x,y)=1, \quad (x,y)\in\tilde{\mathscr{R}}_K \quad \text{and} \quad x,y\in X_0$$

(see [**Sch**, § 2, p. 25]).

LEMMA 3.3. *Let* (X,μ), $\mathscr{R}_K$, G, *and* $\psi(\alpha)$ *be as above. There exists a homomorphism* $\beta:\mathscr{R}_K\to G$, *cohomologous to* $\psi(\alpha)$, *such that for every* $\gamma\in K$, *the map* $x\mapsto\beta(x,x\gamma)$ *takes only finitely many values.*

Proof. First of all let us introduce the following notation:

$C(y_1,\ldots,y_k)$ denotes the cylinder set $\{x\in X;\, x_1=y_1,\ldots,x_k=y_k\}$.

$C(i;n)$ denotes the cylinder set $\{x\in X;\, x_n=i\}$ for $i\in\{0,1\}$ and $n\geq 1$.

K_n denotes the subgroup of K generated by $S_1,\ldots,S_n$.

Let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive numbers, with $0<\varepsilon_n<1$ and $\sum_{n\geq 1}\varepsilon_n<\infty$. We shall construct by induction a sequence $(\varphi_n)_{n\geq 1}$ of measurable functions $\varphi_n:X\to G$ such that

(i) $\mu(\{x\in X;\, \varphi_n(x)\neq e\})\leq\varepsilon_n$,

(ii) if $\alpha_0=\alpha$ and $\alpha_n(x,x\gamma)=\varphi_n(x)\alpha_{n-1}(x,x\gamma)\varphi_n(x\gamma)^{-1}$, then $G_n=\{\alpha_n(x,x\gamma);\, x\in X;\, \gamma\in K_n\}$ is a finite subset of G.

Let φ_0 denote the constant function $\varphi_0(x)=e$ for all $x\in X$. Assume $\varphi_0,\varphi_1,\ldots,\varphi_{n-1}$ built, and let us construct φ_n.

For all $g\in G$, set $F_g=\{x\in C(0;n);\ \alpha(x,xS_n)=g\}$. As $\coprod_{g\in G}F_g=C(0;n)$, there exists a finite set $M\subset G$ such that

$$\mu\left(C(0;n)\Big\backslash\bigcup_{g\in M}F_g\right)\le\frac{\epsilon_n}{2}\quad\text{and}\quad\mu\left(C(1;n)\Big\backslash\bigcup_{g\in M}F_gS_n\right)\le\frac{\varepsilon_n}{2}.$$

Set $F=\bigcup_{\gamma\in K_{n-1}}\bigcup_{g\in M}F_g\gamma$ and consider the map φ_n defined by

$$\varphi_n(x)=\begin{cases}e & \text{if } x\in F\cup FS_n,\\ \alpha_{n-1}(x_0,x_0S_n^j) & \text{if } x\notin F\cup FS_n,\end{cases}$$

where $x=x_0S_n^j$, $x_0\in C(0;n)\backslash F, j\in\{0,1\}$. Then

$$\mu(\{x\in X;\ \varphi_n(x)\neq e\})\le\mu(X\backslash(F\cup FS_n))\le\varepsilon_n,$$

which proves (i).

Note that by definition F is K_{n-1}-invariant and therefore $F\cup FS_n$ is K_n-invariant.

We divide the proof of (ii) into different cases. First of all, let $x\in F\cup FS_n$. If $\gamma\in K_{n-1}$, then $\alpha_n(x,x\gamma)=\alpha_{n-1}(x,x\gamma)\in G_{n-1}$. If $\gamma\in K_n\backslash K_{n-1}$, then $\gamma=\gamma'S_n$ with $\gamma'\in K_{n-1}$. We have $x=x_0\sigma S_n^j$, with $x_0\in\bigcup_{g\in M}F_g$, $\sigma\in K_{n-1}$ and $j\in\{0,1\}$. Then

$$\begin{aligned}\alpha_n(x,x\gamma)&=\alpha_{n-1}(x,x\gamma)\\&=\alpha_{n-1}(x_0\sigma s_n^j,x_0\sigma\gamma'S_n^{j+1})\\&=\alpha_{n-1}(x_0S_n^j\sigma,x_0S_n^j)\alpha_{n-1}(x_0S_n^j,x_0S^{j+1})\alpha_{n-1}(x_0S_n^{j+1}\sigma\gamma').\end{aligned}$$

As σ and $\sigma\gamma'\in K_{n-1}$, we have $\alpha_n(x,x\gamma)\in G_{n-1}^{\pm1}M^{\pm1}G_{n-1}^{\pm1}$.

Now, if $x\notin F\cup FS_n$, then $x=x_0S_n^j$ with $x_0\in C(0;n)\backslash F$. If $\gamma\in K_n$, let us write $\gamma=\gamma'S_n^i$ with $\gamma'\in K_{n-1}$, and $i=0$ or 1. As $x\gamma=x_0\gamma'S_n^{j+i}$, we have $(x\gamma)_0=x_0\gamma'$. Hence,

$$\begin{aligned}\alpha_n(x,x\gamma)&=\varphi_n(x)\alpha_{n-1}(x,x\gamma)\varphi_n(x\gamma)^{-1}\\&=\alpha_{n-1}(x_0,x_0S_n^j)\alpha_{n-1}(x_0S_n^j,X\gamma)\varphi_n(x\gamma)^{-1}\\&=\alpha_{n-1}(x_0,x\gamma)\alpha_{n-1}(x\gamma,x_0\gamma')\\&=\alpha_{n-1}(x_0,x_0\gamma')\in G_{n-1}.\end{aligned}$$

Therefore, (ii) holds.

Note that if $\gamma\in K_{n-1}$, then we have $\alpha_n(x,x\gamma)\in G_{n-1}x\in X$; more generally,

$$\alpha_k(x,x\gamma)\in G_{n-1}\quad\text{for}\quad x\in X\quad\text{and}\quad k\ge n.$$

By (i), the sequence $(\varphi_n(x),\dots,\varphi_1(x))_{n\ge1}$ is a Cauchy sequence for a.e. x. Let $\varphi:X\to G$ denote the limit (see for example [**Sh**, p. 257]) and set

$$\beta(x,x\gamma)=\varphi(x)\alpha(x,x\gamma)\varphi(x\gamma)^{-1}.$$

Then β satisfies the requirements of the Lemma. □

By construction of $(X,\mu,\mathcal{R}_K,\beta)$ and by definition of the Mackey range (see for example [**Z3**, Proposition 4.2.24]), the two G-spaces $\mathrm{MR}(\tilde{S},\tilde{\mu},\tilde{\mathcal{R}},\alpha)$ and $\mathrm{MR}(X,\mu,\mathcal{R}_K,\beta)$ are isomorphic.

Second part of the proof of Proposition 3.2. Before stating Lemma 3.5, let us recall the following definition from [**S3**]:

DEFINITION 3.4. (1) A Bratteli diagram D is a graph with set of vertices V and set of edges E with the following properties:

(a) V is the disjoint union of subsets $V(n)$ $(n\ge0)$ with $|V(n)|<\infty$ for all $n\ge0$.

(b) E is the disjoint union of subsets $E(n)$ $(n \geqslant 0)$, with each edge $e \in E(n)$ connecting a vertex $s(e) \in V(n)$ with a vertex $r(e) \in V(n+1)$.

(c) For every vertex $v \in V$, there exist $e, f \in E$ with $s(e) = v$, $r(f) = v$ (except for $v \in V(0)$, for which we omit the second requirement).

(2) A path in D is a sequence (e_k) of edges with $s(e_0) \in V(0)$, and for $k \geq 1$, $s(e_k) = r(e_{k-1})$.

We denote by Ω_n the space of paths of length n, and by Ω the space of paths of infinite length. We view Ω as a topological space, with basis $\{\Omega(f); f \in \Omega_n,\ n \geq 1\}$, where to each $f = (f_0, f_1, \ldots, f_{n-1}) \in \Omega_n$ we associate the set

$$\Omega(f) = \{e \in \Omega,\ e_k = f_k;\ 0 \leq k \leq n\}.$$

(3) An AF-measure (or Markov measure) μ_p on Ω is a measure determined by a system of transition probabilities p (i.e. maps $p: E \to [0, 1]$ with $p(e) \geq 0$ and for every vertex v $\sum_{\{e, s(e)=v\}} p(e) = 1$), given by

$$\mu_p(\Omega(f)) = \prod_{k=0}^{n} p(f_k),$$

where $f = f(f_0, f_1, \ldots, f_n) \in \Omega_n$.

Note that Ω carries a canonical equivalence relation $\mathcal{R}_\Omega$ defined by

$$e\mathcal{R}_\Omega f \Leftrightarrow \text{for some } n \qquad e_k = f_k \quad \text{for all } k \geq n.$$

LEMMA 3.5. *Let X, μ, $\mathcal{R}_K$, and β be as in Lemma 3.3. Then there exist*

(i) *a Bratteli diagram D, an AF-measure μ_p on the path space Ω of D, and an isomorphism $\psi: (X, \mu) \to (\Omega, \mu_p)$ such that $\psi(\mu) = \mu_p$ and*

$$\psi(x)\mathcal{R}_\Omega\psi(y) \Leftrightarrow x\mathcal{R}_K y,$$

(ii) *a sequence $(b_n)_{n \geq 0}$ of maps $b_n: E(n) \to G$ such that if $\beta' \in Z^1(\mathcal{R}_\omega, G)$ denotes the cocycle given by*

$$\beta'(e, f) = b_0(e_0) \ldots b_n(e_n) b_n(f_n)^{-1} \ldots b_0(f_0)^{-1}$$

whenever $e\mathcal{R}_\Omega f$ and $e_k = f_k$ for $k \geq n+1$, then

$$\beta(x, xS_k) = \beta'(\psi(x), \psi(xS_k)) \quad \mu\text{-a.e.}, k \in \mathbb{N}.$$

Proof. By Lemma 3.3, for each $n \geq 1$, there exists a measurable partition $(A_j^n)_{1 \leq j \leq k_n}$ of $C(y_1 = 0, \ldots, y_n = 0) \subset X$ such that for $1 \leq j \leq k_n$, and $x \in A_j^n$,

$$\frac{d\mu S_n}{d\mu}(x) = a_{n,j} \quad \text{and} \quad \beta(x, xS_n) = g_{n.j}.$$

Set $\mathcal{B}_0 = \{X\}$. By induction, we define a sequence $(\mathcal{B}_n)_{n \geq 0}$ of measurable partitions (modulo μ-null sets) of $C(y_1 = 0, \ldots, y_n = 0)$ by

$$\mathcal{B}_n = \{B \cap B'S_n \cap A_j^n;\ B, B' \in \mathcal{B}_{n-1}, 1 \leq j \leq k_{n-1}, \quad \text{and} \quad \mu(B \cap B'S_n \cap A_j^n) \neq 0\}.$$

To this sequence $(\mathcal{B}_n)_{n \geq 0}$, we associate the following Bratteli diagram D.

For each $n \geq 0$, the set of vertices $V(n)$ is equal to $\mathcal{B}_n$.

Let $C = B \cap B'S_{n+1} \cap A_j^{n+1} \in \mathcal{B}_{n+1}$. Then there are two edges e in $E(n)$ with range C: the edge $e(B, C)$ corresponding to the inclusion of C in $B \in \mathcal{B}_n$, and the edge

$e(B', C)$, corresponding to $CS_{n+1} \subset B' \in \mathcal{B}_n$. Thus, $s(e(B, C)) = B$, $s(e(B', C)) = B'$, and $r(e(B, C)) = r(e(B', C)) = C$.

Define a map $p: E(n) \to [0, 1]$ by

$$p(e(B, B \cap B'S_{n+1} \cap A_j^{n+1})) = \frac{\mu(B \cap B'S_{n+1} \cap A^{n+1})}{\mu(B)}$$

and

$$p(e(B, B' \cap BS_{n+1} \cap A_j^{n+1}) = \frac{\mu(B \cap B'S_{n+1} \cap A_j^{n+1}S_{n+1})}{\mu(B)}.$$

For each $B \in \mathcal{B}_n = V(n)$, we have

$$\sum_{\{e \in E(n); s(e) = B\}} p(e)$$

$$= \sum_{\substack{B' \in \mathcal{B}_n \\ 1 \le l \le k_{n+1}}} \frac{\mu(B \cap B'S_{n+1} \cap A_j^{n+1}) + \mu(B'S_{n+1} \cap B \cap A_j^{n+1}S_{n+1})}{\mu(B)}$$

$$= \sum_{B' \in \mathcal{B}_n} \frac{\mu(B \cap B'S_{n+1} \cap C(y_1 = 0, \ldots, y_{n+1} = 0))}{\mu(B)}$$

$$+ \sum_{B' \in \mathcal{B}_n} \frac{\mu(B'S_{n+1} \cap B \cap C(y_1 = 0, \ldots, y_n = 0, y_{n+1} = 1))}{\mu(B)}$$

$$= \frac{\mu(B \cap C(y_1 = 0, \ldots, y_{n+1} = 0))}{\mu(B)}$$

$$+ \frac{\mu(B \cap C(y_1 = 0, \ldots, y_n = 0, y_{n+1} = 1))}{\mu(B)}$$

$$= 1.$$

Therefore, $p: E = \coprod_{n \ge 0} E(n) \to [0, 1]$ is a system of transition probabilities on D. Denote by μ_p the corresponding AF-measure on the path space Ω of D.

Let $n \ge 1$. If $(f_0, \ldots, f_{n-1}) \in \Omega_n$, then by the definition of the edges of D, for each $1 \le k \le n$ there are elements $B_k \in \mathcal{B}_k$ such that

$$B_{k+1} S_{k+1}^{\varepsilon_{k+1}} \subset B_k.$$

Hence we get the following chain of inclusions:

$$X \supset B_1 S_1^{\varepsilon_1} \supset B_2 S_2^{\varepsilon_2} S_1^{\varepsilon_1} \supset \cdots \supset B_n S_n^{\varepsilon_n} \cdots S_1^{\varepsilon_1}.$$

Set $\theta_n(\Omega(f_0, \ldots, f_{n-1})) = B_n S_n^{\varepsilon_n} \ldots S_1^{\varepsilon_1}$. Then θ_n establishes a bijective correspondence between

$$\Omega_n = \{\Omega(f); f \in \Omega_n\}$$

and

$$\tilde{\mathcal{B}}_n = \{B_n S_n^{\varepsilon_n} \ldots S_1^{\varepsilon_1};\ B_n \in \mathcal{B}_n,\ \varepsilon_i \in \{0, 1\},\ 1 \le i \le n\}.$$

Indeed, suppose that $B_nS_n^{\varepsilon_n}\dots S_1^{\varepsilon_1}\in\tilde{\mathscr{B}}_n$. Then by definition of the partitions $\mathscr{B}_k$, there exists for each $1\le k\le n+1$ one and only one element $B_k\in\mathscr{B}_k$ such that

$$B_nS_n^{\varepsilon_n}\dots S_1^{\varepsilon_1}\subset B_{n-1}S_{n-1}^{\varepsilon_{n-1}}\dots S_1^{\varepsilon_1}\subset\cdots\subset B_1S_1^{\varepsilon_1}\subset X.$$

These inclusions determine a unique path $(f_0,\dots,f_{n-1})\in\Omega_n$, where $f_i=e(B_i,B_{i+1})$ corresponds (cf. above) to the inclusion

$$B_{i+1}S_{i+1}^{\varepsilon_{i+1}}\subset B_i.$$

For all $n\ge m\ge 1$, $\tilde{\mathscr{B}}_m\subset\tilde{\mathscr{B}}_n$ and $\Omega(m)\subset\Omega(n)$; moreover, θ_n extends θ_m.

If $B_k\in\mathscr{B}_k$, then for $1\le j\le k$, $d\mu S_k/d\mu(x)$ is constant on B_k. Therefore, we have

$$\mu(B_nS_n^{\varepsilon_n},\dots,S_1^{\varepsilon_1})=\prod_{i=0}^{n-1}\frac{\mu(B_{i+1}S_{i+1}^{\varepsilon_{i+1}}\dots S_1^{\varepsilon_1})}{\mu(B_iS_i^{\varepsilon_i}\dots S_1^{\varepsilon_1})}=\prod_{i=0}^{n}\frac{\mu(B_{i+1}S_{i+1}^{\varepsilon_{i+1}})}{\mu(B_i)}$$

$$=\prod_{i=0}^{n}p(f_i)=\mu_p(\Omega(f))=\mu_p(\theta_n(B_nS_n^{\varepsilon_n}\dots S_1^{\varepsilon_1})).$$

As $\bigcup_{n\ge1}\tilde{\mathscr{B}}_n$ (resp. $\bigcup_{n\ge1}\Omega(n)$) spans the σ-algebra of (X,μ), modulo μ-null sets (resp. the σ-algebra of (Ω,μ_p), modulo μ_p-null sets), the family $(\theta_n)_{n\ge1}$ gives rise to an isomorphism

$$\psi:(X,\mu)\to(\Omega,\mu_p)$$

such that $\psi(\mu)=\mu_p$ and for all $n\ge1$, $\psi^*=\theta_n$. The description of ψ is as follows: For a.e. $x\in X$, there exists a sequence $(B_j,\varepsilon_j)_{j\ge1}$ with $B_j\in\mathscr{B}_j$ and $\varepsilon_j\in\{0,1\}$ such that

$$x\in\cdots\subset B_kS_k^{\varepsilon_k}\cdots S_1^{\varepsilon_1}\subset B_{k-1}S_{k-1}^{\varepsilon_{k-1}}\cdots S_1^{\varepsilon_1}\subset\cdots\subset B_1S_1^{\varepsilon_1}.$$

This chain of inclusions defines the path $\psi(x)=(e_n)_{n\ge0}\in\Omega$, where e_n corresponds to the inclusion $B_{n+1}S_{n+1}^{\varepsilon_{n+1}}\subset B_n$.

Let k be given. Then for μ-a.e. $x\in X$ there exist two uniquely defined sequences $(B_j,\varepsilon_j)_{j\ge1}$ and $(C_j,\delta_j)_{j\ge1}$ with

(a) $\quad B_j, C_j\in\mathscr{B}_j$ and $\varepsilon_j,\delta_j\in\{0,1\}$ for $j\ge1$,

(b) $\quad x\in\cdots\subset B_kS_k^{\varepsilon_k}\cdots S_1^{\varepsilon_1}\subset B_{k-1}S_{k-1}^{\varepsilon_{k-1}},\dots,S_1^{\varepsilon_1}\subset\cdots\subset B_1S_1^{\varepsilon_1}\subset X,\qquad$ (3.5.1)

(c) $\quad xS_k\in\cdots\subset C_jS_j^{\delta_j}\cdots S_1^{\delta_1}\subset\cdots\subset C_1S_1^{\delta_1}\subset X.$

Note that for $n\ge k$, if $B\in\tilde{\mathscr{B}}_n$, then $BS_k\in\tilde{\mathscr{B}}_n$. Then we have:

$$\begin{aligned}&\delta_j=\varepsilon_j\quad\text{for}\quad j\ne k\quad\text{and}\quad\delta_k=\varepsilon_k+1\ (\text{modulo }2);\\&C_j=B_j\quad\text{for}\quad j\ge k;\\&\text{for } j<k,\ C_j \text{ is the element of } \mathscr{B}_j \text{ containing } C_{j+1}S_{j+1}^{\delta_{j+1}}.\end{aligned}\qquad(3.5.2)$$

For $j\ge0$, let e_j (resp. f_j) denote the edge corresponding to the inclusion $B_{j+1}S_{j+1}^{\varepsilon_{j+1}}\subset B_j$ (resp. $C_{j+1}S_{j+1}^{\delta_{j+1}}\subset C_j$). Then $\psi(x)=(e_j)_{j\ge0}$ and $\psi(xS_k)=(f_j)_{j\ge0}$. By (3.5.2), $e_j=f_j$ for $j\ge k$.

This shows that ψ intertwines the equivalence relation $\mathcal{R}_K$ on (X, μ) and $\mathcal{R}_\Omega$ on (Ω, μ_p), and finishes the proof of (i)

Proof of (*ii*). For each $n \geq 0$, define a map $b_n: E(n) \to G$ by

$$b_n(e(B, B \cap B'S_{n+1} \cap A_j^{n+1})) = 1,$$
$$b_n(e(B, BS_{n+1} \cap B' \cap A_j^{n+1}S_{n+1})) = g_{n+1,j}^{-1}.$$

With β' defined with respect to this sequence for fixed k we have

$$\beta'(\psi(x), \psi(xS_k)) = b_0(e_0), \ldots, b_{k-1}(e_{k-1})b_{k-1}(f_{k-1})^{-1} \cdots b_0(f_0)^{-1} \quad \text{a.e. } x \in X.$$

To compute $\beta(x, xS_k)$, set $y_0 = x$, and set $y_i = xS_1^{\varepsilon_1} \cdots S_i^{\varepsilon_i}$, $i \geq 1$. We have

$$\beta(x, x, S_k) = \beta(y_0, y_1)\beta(y_1, y_2) \cdots \beta(y_{k-1}, y_k)\beta(y_k, y_kS_k)\beta(y_kS_k, y_{k-1}S_k) \cdots \beta(y_1S_k, y_0S_k).$$

For $1 \leq i \leq k$, by (3.5.1)(b) we have that $y_i \in B_i$. With $j(i)$ such that $B_i \subset A_{j(i)}^i$, we have

$$\beta(y_{i-1}, y_i) = \beta(y_iS_i^{\varepsilon_i}, y_i) = \begin{cases} 1 & \text{if } \varepsilon_i = 0, \\ g_{i,j(i)}^{-1} & \text{if } \varepsilon_i = 1. \end{cases}$$

As e_{i-1} is the edge corresponding to $B_iS_i^{\varepsilon_i} \subset B_{i-1}$, it follows that

$$\beta(y_{i-1}, y_i) = b_{i-1}(e_{i-1}). \tag{3.5.4}$$

For $1 \leq i \leq k-1$, by 3.5.1(c), we have $y_iS_k \in C_i$ and $y_{i-1}S_k = y_iS_i^{\varepsilon_i}S_k \in C_{i-1}$. As f_{i-1} is the edge corresponding to the inclusion $C_iS_i^{\varepsilon_i} \subset C_{i-1}$, we have

$$\beta(y_{i-1}S_k, y_iS_k) = b_{i-1}(f_{i-1}). \tag{3.5.5}$$

For $i = k$, let us assume first that $\varepsilon_k = 0$. Then $B_k = B_{k-1} \cap C_{k-1}S_k \cap A_{j(k)}^k$ and $y_k = y_{k-1}S_k^{\varepsilon_k} = y_{k-1}$. Hence,

$$\beta(y_k, y_kS_k) = b_{k-1}(f_{k-1})^{-1},$$

because $y_kS_k \in C_{k-1}$ and f_k is the edge corresponding to the inclusion of $B_kS_k \subset C_{k-1}$; hence,

$$\beta(y_kS_k, y_{k-1}S_k) = \beta(y_{k-1}S_k, y_{k-1}S_k) = 1.$$

In other words,

$$\beta(y_k, y_kS_k)\beta(y_kS_k, y_{k-1}S_k) = b_{k-1}(f_{k-1})^{-1}. \tag{3.5.6}$$

If $\varepsilon_k = 1$, then (3.5.6) is verified in a similar way. From (3.5.3) to (3.5.6), (ii) follows. □

Let D be the Bratteli diagram, μ_p the AF-measure on the path space of D, and $\beta': \mathcal{R}_\Omega \to G$ the homomorphism obtained by Lemma 3.5.

For each $n \geq 0$ and each $(v, w) \in V(n) \times V(n+1)$, consider the positive measure $\sigma_{n,v,w}$ on G given by

$$\sigma_{n,v,w} = \sum_{\{e \in E(n); s(e) = v, r(e) = w\}} p(e)\delta_{b_n(e)}$$

where $\delta_{b_n(e)}$ is the Dirac function at $g = b_n(e)$. Note that, for every $v \in G$,

$$\sum_{w\in V(n+1)} \sigma_{n,v,w}(G) = \sum_{w\in V(n+1)} \sum_{\{e\in E(n); s(e)=v, r(e)=w\}} p(e) = 1.$$

For each $n \geq 0$, set $F_n = \coprod_{v\in V(n)} G$ and consider the transition probability P^n_{n+1} from F_n to F_{n+1} given by

$$P^n_{n+1}((v, g), (w, A)) = \sigma_{n,v,w}(g^{-1}A).$$

Then by [**S3**, Theorem 2], the G-space of harmonic functions of this matrix-valued random walk on G is isomorphic (as a G-space) with the Mackey range $\mathrm{MR}(\Omega, \mu_p, \beta')$. Therefore, Proposition 3.2 is proved.

4. *The main theorem*

We finish the proof of the following result.

THEOREM 4.1. *Let S be a standard Borel space with a non-atomic probability measure μ and G be a (countable) discrete group acting ergodically on (S, μ). The following four statements are equivalent*:

(i) *(S, μ) is an amenable G-space*;

(ii) *the equivalence relation induced by the action of G on (S, μ) is amenable and the stabilizer subgroup $G_s = \{g \in G; sg = s\}$ is amenable μ-a.e.*

(iii) *(S, μ) is isomorphic as a G-space to the Mackey range of a homomorphism from an amenable ergodic equivalence relation to G*;

(iv) *(S, μ) is isomorphic as a G-space to the Poisson boundary of a group-invariant matrix-valued Markov random walk on G.*

Proof. Let (S, μ) be an amenable G-space. Then the equivalence relation $\mathcal{R}_G$ is amenable by [**Z2**, Proposition 3.4]. Moreover, by Proposition 1.1, the stability subgroups G_s are amenable, μ-a.e. Therefore (i) implies (ii).

The implications (ii) to (iii) and from (iii) to (iv) are just Theorems 2.1 and 3.1. In [**CW**, Remark 2.3], A. Connes and E. J. Woods state the implication from (iv) to (i). A detailed proof can be found in [**J**]. □

Remark 4.2.

(a) The implication from (ii) to (i) is stated without proof as Theorem 4 of [**H2**].

(b) In [**Z1**, Theorem 3.3] R. J. Zimmer proves that (ii) implies (i), and also that (iii) implies (i).

(c) An alternative proof that (iv) implies (i) can be given by providing directly the implication from (iv) to (iii). Let us give a sketch of this proof.

Let (B, μ) be a G-space which is the Poisson boundary of a right group-invariant random walk $\mathcal{R}$ on G, given by a sequence $\{\sigma_n\}_{n\geq 1}$ of $(l_{n-1} \times l_n)$-matrices of positive measures on G satisfying 3.1. Using a generalization of Lemma 2.5 of [**CW**], we may assume that for all $n \geq 1$, the cardinal $k_{n,i,j}$ of Supp $(\sigma_{n,i,j})$ is finite for $1 \leq i \leq l_{n-1}$, $1 \leq j \leq l_n$. Let us write

$$\mathrm{Supp}\,(\sigma_{n,i,j}) = \{b^b_{i,j}(1), \ldots, b^n_{i,j}(k_{n,i,j})\}.$$

Then consider the Bratteli diagram D defined (with the notation of Definition 3.4) by:

$$V(n)=\{v_1^n,\ldots,v_{l_n}^n\} \text{ for } n\geq 0,$$

$$\text{for } 1\leq i\leq l_{n-1} \quad \text{and} \quad 1\leq j\leq l_n$$

$$\text{the set } \{e(b_{i,j}^n(k))\}_{1\leq k\leq k_{n,i,j}} \text{ denotes the edges from } v_i^{n-1}$$

Consider the map $p: E=\coprod_{n\geq 0} E(n)\to[0,1]$ given by

$$p(e(b_{n,i,j}(k)))=\sigma_{n,i,j}(b_{i,j}^n(k)) \quad \text{for } 1\leq k\leq k_{n,i,j}.$$

By equation (3.1), for $1\leq i\leq l_{n-1}$ and $1\leq j\leq l_n$, we have

$$\sum_{\{e\in E(n);s(e)=v_i^{n-1}\}} p(e)=\sum_{j=1}^{l_n}\sum_{k=1}^{k_{n,i,j}}\sigma_{n,i,j}(b_{i,j}^n(k))=1.$$

Therefore, p is a system of transition probabilities on D.

For each $n\geq 0$, we consider the map $b_n: E(n)\to G$ defined by

$$b_n(e(b_{i,j}^n(k)))=b_{i,j}^n(k).$$

Let $h=(h_n)_{n\geq 0}\in\mathcal{H}^\infty$ be a (bounded) harmonic function associated with $\mathcal{R}$. As $E_n=V(n)\times G$, we have

$$\begin{aligned} h_{n-1}(i,g)=h_{n-1}(v_i^{n-1},g)&=\int_{E_n} h_n((j,x))P_n^{n-1}((i,g),d(j,x)) \\ &=\sum_{j=1}^{l_n}\int_G h_n((j,x))\,d\sigma_{n,i,j}(g^{-1}x)=\sum_{j=1}^{l_n}\int_G h_n(j,gx)\,d\sigma_{ni,i,j}(x) \\ &=\sum_{j=1}^{l_n}\sum_{k=1}^{k_{n,i,j}} h_n(v_j^n,gb_{i,j}^n(k))p(e(b_{i,j}^n(k))) \\ &=\sum_{\{e;s(e)=v_i^{n-1}\}} p(e)h_n(r(e),gb_n(e)). \end{aligned}$$

On the path space Ω of D, let μ_p denote the AF-measure determined by the system of transition probabilities p and let $\mathcal{R}_\Omega$ denote the canonical equivalence relation on Ω (see 3.4).

Let $\beta\in Z^1(\mathcal{R}_\omega,G)$ denote the cocycle given by

$$\beta(e,f)=b_0(e_0)\cdots b_n(e_n)b_n(f_n)^{-1}\cdots b_0(f_0)^{-1}$$

whenever $e\mathcal{R}_\Omega f$ and e_k for $k\geq n+1$.

By [**S3**, Theorem 2], there is a natural G-isomorphism between the Mackey range $\mathrm{MR}((\Omega,\mu_p),\mathcal{R}_\Omega,\beta)$ and the Poisson boundary of the matrix-valued random walk $\mathcal{R}$.

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