WITT GROUPS OF THE PUNCTURED SPECTRUM OF A 3-DIMENSIONAL REGULAR LOCAL RING AND A PURITY THEOREM

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0. *Introduction*

Let A be a regular local ring with quotient field K. Assume that 2 is invertible in A. Let $W(A) \longrightarrow W(K)$ be the homomorphism induced by the inclusion $A \hookrightarrow K$, where $W(\,)$ denotes the Witt group of quadratic forms. If $\dim A \leqslant 4$, it is known that this map is injective [**6**, **7**]. A natural question is to characterize the image of $W(A)$ in $W(K)$. Let $\mathrm{Spec}^1(A)$ be the set of prime ideals of A of height 1. For $P \in \mathrm{Spec}^1(A)$, let π_P be a parameter of the discrete valuation ring A_P and $k(P) = A_P/PA_P$. For this choice of a parameter π_P, one has the *second residue homomorphism* $\partial_P : W(K) \longrightarrow W(k(P))$ [**9**, p. 209]. Though the homomorphism ∂_P depends on the choice of the parameter π_P, its kernel and cokernel do not. We have a homomorphism

$$\partial = (\partial_P) : W(K) \longrightarrow \bigoplus_{P \in \mathrm{Spec}^1(A)} W(k(P)).$$

A part of the so-called Gersten conjecture is the following question on 'purity'. Is the sequence

$$W(A) \longrightarrow W(K) \xrightarrow{\partial} \bigoplus_{P \in \mathrm{Spec}^1(A)} W(k(P))$$

exact? This question has an affirmative answer for $\dim(A) \leqslant 2$ [**1**; **3**, p. 277]. There have been speculations by Pardon and Barge-Sansuc-Vogel on the question of purity. However, in the literature, there is no proof for purity even for $\dim(A) = 3$. One of the consequences of the main result of this paper is an affirmative answer to the purity question for $\dim(A) = 3$.

We briefly outline our main result. For any scheme X let $W^\epsilon(X)$ denote the Witt group of ϵ-symmetric spaces on X, $\epsilon = \pm 1$ ($W^{+1}(X) = W(X)$ being the usual Witt group of symmetric spaces over X). Let A be a regular local ring of dimension 3 with maximal ideal m and $Y = \mathrm{Spec}(A) \backslash \{m\}$. We associate (§3) to an ϵ-symmetric space over Y a $(-\epsilon)$-symmetric space over a finite-length A-module. This assignment leads to a homomorphism $W^\epsilon(Y) \longrightarrow W^{-\epsilon}_{\mathrm{fl}}(A)$, where $W^\epsilon_{\mathrm{fl}}(A)$ is the Witt group of ϵ-symmetric spaces of finite-length A-modules (cf. §1). Then we prove (§4) that the sequence

$$0 \longrightarrow W^\epsilon(A) \longrightarrow W^\epsilon(Y) \longrightarrow W^{-\epsilon}_{\mathrm{fl}}(A) \longrightarrow 0$$

is exact, where the map $W^\epsilon(A) \longrightarrow W^\epsilon(Y)$ is induced by the restriction. Since $W^\epsilon_{\mathrm{fl}}(A) \simeq W^\epsilon(A/m)$, it follows that $W^{-1}_{\mathrm{fl}}(A) = 0$. Thus the map $W(A) \longrightarrow W(Y)$ is an isomorphism. This leads to the purity theorem for the Witt groups. On the other hand, since every skew-symmetric space over A is hyperbolic, $W^{-1}(A) = 0$ and we get an isomorphism $W^{-1}(Y) \simeq W(A/m)$. We observe the curious fact that if A is complete, $W^{\pm 1}(Y)$ is isomorphic to $W(A/m)$.

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A crucial result used in our proof of the main theorem is a theorem of Horrocks [**2**] on vector bundles on the punctured spectrum $Y = \mathrm{Spec}(A)\backslash\{m\}$, where A is a regular local ring of dimension 3 and m is its maximal ideal. We use his theorem on the equivalence of the category of 'Φ-equivalence' classes of vector bundles on Y with the category of finite-length A-modules.

We would like to remark parenthetically that purity for dimension 3 was used in [**8**] while establishing the equivalence of the finite generation of Witt groups of affine real 3-folds and the finite generation of Chow groups of codimension 2 cycles modulo 2.

1. *ϵ-symmetric spaces reminisced*

Let A be a regular local ring of dimension 3 in which 2 is invertible. We recall the definition of ϵ-symmetric spaces on finite-length A-modules and their Witt groups. For A-modules M, N and $i \geqslant 0$, let $\mathrm{Ext}^i(M,N)$ denote the group of congruence classes of i-fold extensions of N by M [**4**, p. 84]. For any homomorphism $f\colon M \longrightarrow M'$ of A-modules, let $\mathrm{Ext}^i(N,f)\colon \mathrm{Ext}^i(N,M) \longrightarrow \mathrm{Ext}^i(N,M')$ be the induced homomorphisms defined as follows. Let

$$\zeta = 0 \longrightarrow M \xrightarrow{\alpha} Z_i \xrightarrow{\partial_i} Z_{i-1} \longrightarrow \cdots \longrightarrow Z_2 \xrightarrow{\partial_2} Z_1 \xrightarrow{\beta} N \longrightarrow 0$$

be an i-fold extension of N by M. Let $Z = (Z_i \oplus M')/(\{(\alpha(x), f(x)) \mid x \in M\})$ be the push-out of Figure 1 [**4**].

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & Z_i \\ \big\downarrow{\scriptstyle f} & & \\ M' & & \end{array}$$

FIGURE 1.

Then

$$\mathrm{Ext}^i(N,f)(\zeta) = 0 \longrightarrow M' \xrightarrow{\alpha'} Z \xrightarrow{\partial'} Z_{i-1} \xrightarrow{\partial_{i-1}} \cdots \longrightarrow Z_2 \xrightarrow{\partial_2} Z_1 \xrightarrow{\beta} N \longrightarrow 0$$

where α' and ∂' are the natural homomorphisms induced by the push-out. Similarly, we define $\mathrm{Ext}^i(f,N)$ as the pull-back under f of an i-fold extension of N by M'. Let M be a finite-length A-module and $M^\vee = \mathrm{Ext}^3(M,A)$. If M, M' are two finite-length A-modules and $f\colon M \longrightarrow M'$ is an A-linear map, then we denote $\mathrm{Ext}^3(f,A)$ by $f^\vee$. Let

$$\mathscr{P} = 0 \longrightarrow P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\theta} M \longrightarrow 0$$

be a projective resolution of M. Since $\mathrm{Ext}^i(M,A) = 0$ for $i = 0,1,2$ [**5**, Theorem 18.1], by dualizing the above exact sequence we see that

$$\mathscr{P}^* = 0 \longrightarrow P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\partial_2^*} P_2^* \xrightarrow{\partial_3^*} P_3^* \xrightarrow{\theta'} M^\vee \longrightarrow 0$$

is a projective resolution of $M^\vee$, where $P_i^* = \mathrm{Hom}_A(P_i, A)$, ∂_i^* is induced by ∂_i and for any $f \in P_3^*$,

$$\theta'(f) = \mathrm{Ext}^3(f,M)(\mathscr{P}) \in M^\vee.$$

Throughout this paper, for any surjection $\theta: P_0 \longrightarrow M$ as above, θ' denotes the map defined as above. We define a canonical homomorphism $\mathscr{C}: M \longrightarrow M^{\vee\vee}$ as follows. Let $x \in M$. Choose $y \in P_0$ such that $\theta(y) = x$. We define

$$\mathscr{C}(x) = \mathrm{Ext}^3(-e_y, M^{\vee})(\mathscr{P}^*) \in M^{\vee\vee},$$

where, for $f \in P_0^*$, $e_y(f) = f(y)$. Then it is easy to see that $\mathscr{C}(x)$ is independent of the choice of y and Figure 2 is commutative, where $\mathscr{C}: P_i \longrightarrow P_i^{**}$ are the canonical isomorphisms.

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & P_3 & \xrightarrow{\partial_3} & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\theta} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathscr{C}} & & \downarrow{\scriptstyle \mathscr{C}} & & \downarrow{\scriptstyle \mathscr{C}} & & \downarrow{\scriptstyle \mathscr{C}} & & \downarrow{\scriptstyle -\mathscr{C}} & & \\
0 & \longrightarrow & P_3^{**} & \xrightarrow{\partial_3^{**}} & P_2^{**} & \xrightarrow{\partial_2^{**}} & P_1^{**} & \xrightarrow{\partial_1^{**}} & P_0^{**} & \xrightarrow{\theta''} & M^{\vee\vee} & \longrightarrow & 0
\end{array}$$

FIGURE 2.

Thus $\mathscr{C}: M \longrightarrow M^{\vee\vee}$ is an isomorphism and it is obvious that it is independent of the choice of the projective resolution. We use this isomorphism to identify M with $M^{\vee\vee}$. The choice of the negative sign at e_y in the definition of $\mathscr{C}$ is explained in the following. Let $m = (x_1, x_2, x_3)$ be the maximal ideal of A and

$$\zeta = 0 \longrightarrow A \xrightarrow{\delta_3} A^3 \xrightarrow{\delta_2} A^3 \xrightarrow{\delta_1} A \xrightarrow{\eta} A/m \longrightarrow 0$$

be the Koszul resolution of A/m with respect to (x_1, x_2, x_3). With respect to the standard basis $\{e_1, e_2, e_3\}$ of A^3, we have

$$\delta_1 = (x_1 \quad x_2 \quad x_3), \quad \delta_2 = \begin{pmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix}$$

and $\eta: A \longrightarrow A/m$ is the natural homomorphism. Let M be a finite-dimensional vector space over A/m. Then M is a finite-length A-module. Let $\tilde{M} = \mathrm{Hom}(M, A/m)$. The assignment $f \mapsto \mathrm{Ext}^3(f, A)(\zeta) \in M^{\vee}$ induces a homomorphism

$$\Phi_M: \tilde{M} \longrightarrow M^{\vee}.$$

The following lemmas are well known, but for the sake of completeness we give their proofs here.

LEMMA 1.1. *The homomorphism Φ_M is an isomorphism and Figure 3 is commutative, where $\iota: M \longrightarrow \tilde{\tilde{M}}$ is the canonical isomorphism.*

$$\begin{array}{ccc}
M & \xrightarrow{\mathscr{C}} & M^{\vee\vee} \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \Phi_M^{\vee}} \\
\tilde{\tilde{M}} & \xrightarrow{\Phi_{\tilde{M}}} & (\tilde{M})^{\vee}
\end{array}$$

FIGURE 3.

Proof. Since $M \simeq \bigoplus_1^n A/m$, $M^\vee \simeq \bigoplus_1^n (A/m)^\vee$ and $\tilde{M} \simeq \bigoplus_1^n \widetilde{A/m}$, it is enough to prove the lemma in the case when $M = A/m$. In this case it is easy to see that $\Phi_M \neq 0$. Since $M^\vee \simeq A/m$ [**5**, Theorem 18.1] and $\tilde{M} \simeq A/m$, Φ_M is an isomorphism. We now prove the commutativity of Figure 3. For all $x \in M$ and $f \in \tilde{M}$ we have $\iota(x)(f) = f(x)$ and

$$\Phi_{\tilde{M}}(\iota(x)) = 0 \longrightarrow A \xrightarrow{\delta_3} A^3 \xrightarrow{\delta_2} A^3 \xrightarrow{\delta_1} A \xrightarrow{\iota(x)^{-1}\eta} A/m \longrightarrow 0.$$

Let $y \in A$ be such that $\eta(y) = x$. Then we have

$$\mathscr{C}(x) = 0 \longrightarrow A \xrightarrow{-\delta_1^* e_y^{-1}} A^{3*} \xrightarrow{\delta_2^*} A^{3*} \xrightarrow{\delta_3^*} A^* \xrightarrow{\eta'} (A/m)^\vee \longrightarrow 0.$$

Since Φ_M is an isomorphism, we have

$$\Phi_M^\vee(\mathscr{C}(x)) = 0 \longrightarrow A \xrightarrow{-\delta_1^* e_y^{-1}} A^{3*} \xrightarrow{\delta_2^*} A^{3*} \xrightarrow{\delta_3^*} A^* \xrightarrow{\Phi_M^{-1}\eta'} (A/m)^\vee \longrightarrow 0.$$

Let $\{e_1^*, e_2^*, e_3^*\}$ be the dual basis of A^{3*}. For $i = 1, 2$, let $\theta_i : A^3 \longrightarrow A^{3*}$ be given by the following matrices, with respect to the bases $\{e_1, e_2, e_3)$ and $\{e_1^*, e_2^*, e_3^*\}$.

$$\theta_1 = \begin{pmatrix} 0 & 0 & -y^{-1} \\ 0 & y^{-1} & 0 \\ -y^{-1} & 0 & 0 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 & 0 & y^{-1} \\ 0 & -y^{-1} & 0 \\ y^{-1} & 0 & 0 \end{pmatrix}.$$

Let $\theta_3 : A \longrightarrow A^*$ be the homomorphism defined by $\theta_3(1) = l_y$, where $l_y(a) = ay$ for all $a \in A$. It is easy to see that Figure 4 is commutative. Thus $\Phi_{\tilde{M}}\,\iota = \Phi_M^\vee \mathscr{C}$.

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A & \xrightarrow{\delta_3} & A^3 & \xrightarrow{\delta_2} & A^3 & \xrightarrow{\delta_1} & A & \xrightarrow{\iota(x)^{-1}\eta} & (A/m)^\vee & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_3} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & A & \xrightarrow{-\delta_1^* e_y^{-1}} & A^{3*} & \xrightarrow{\delta_2^*} & A^{3*} & \xrightarrow{\delta_3^*} & A^* & \xrightarrow{\Phi_M^{-1}\eta'} & (A/m)^\vee & \longrightarrow & 0
\end{array}$$

FIGURE 4. □

LEMMA 1.2. *Let $\psi : M \longrightarrow \tilde{M}$ be a homomorphism and $\tilde{\psi} : \tilde{\tilde{M}} \longrightarrow \tilde{M}$ be the induced homomorphism. Then Figure 5 is commutative.*

$$\begin{array}{ccc}
\tilde{\tilde{M}} & \xrightarrow{\Phi_{\tilde{M}}} & (\tilde{M})^\vee \\
\downarrow{\scriptstyle \tilde{\psi}} & & \downarrow{\scriptstyle \psi^\vee} \\
\tilde{M} & \xrightarrow{\Phi_M} & M^\vee
\end{array}$$

FIGURE 5.

Proof. Let $f \in \tilde{\tilde{M}}$. Then $\Phi_{\tilde{M}}(f)$ is the pull-back of the Koszul resolution ζ under $f : \tilde{M} \longrightarrow A/m$ and $\psi^{\vee}(\Phi_{\tilde{M}}(f))$ is the pull-back of the extension $\Phi_{\tilde{M}}(f)$ under ψ. Thus $\psi^{\vee}(\Phi_{\tilde{M}}(f))$ is the pull-back of the Koszul resolution under the homomorphism $f\psi : M \longrightarrow A/m$. Since $\tilde{\psi}(f) = f\psi$, $\Phi_M(\tilde{\psi})(f)$ is the pull-back of the Koszul resolution under $f\psi$. Thus $\Phi_M \psi^{\vee} = \psi^{\vee} \Phi_{\tilde{M}}$. □

Lemmas 1.1 and 1.2 enable us to embed the category of ϵ-symmetric spaces on finite-dimensional A/m-vector spaces into the category of ϵ-symmetric spaces on finite-length A-modules (cf. Corollary 1.3)).

Let $\epsilon = \pm 1$. An *ϵ-symmetric space of finite length* is a pair (M, ψ) where M is a finite-length A-module and $\psi : M \longrightarrow M^{\vee} = \mathrm{Ext}^3(M, A)$ is an isomorphism with $\psi^{\vee} \mathscr{C} = \epsilon\psi$. Let ψ_1 and ψ_2 be two ϵ-symmetric spaces on finite-length A-modules M_1 and M_2 respectively. We say that ψ_1 is *isometric to* ψ_2 if there exists a homomorphism $\theta : M_1 \longrightarrow M_2$ such that $\psi_1 = \theta^{\vee} \psi_2 \theta$. An ϵ-symmetric space ψ on M is called *metabolic* if there exists a submodule N of M such that

$$0 \longrightarrow N \xrightarrow{\ i\ } M \xrightarrow{\ i^{\vee}\psi\ } N^{\vee} \longrightarrow 0$$

is exact, where $i : N \longrightarrow M$ is the inclusion. The *Witt group* of ϵ-symmetric spaces of finite-length A-modules is defined as the quotient of the Grothendieck group of isometry classes of ϵ-symmetric spaces with the orthogonal sum as addition, modulo the subgroup generated by metabolic spaces. It is denoted by $W^{\epsilon}_{\mathrm{fl}}(A)$.

COROLLARY 1.3. *Let M be a finite-dimensional vector space over A/m. Let $\psi : M \longrightarrow M$ be an ϵ-symmetric space, that is, $\tilde{\psi}\imath = \psi$ and ψ is an isomorphism. Then $\Phi_M \psi : M \longrightarrow M^{\vee}$ is an ϵ-symmetric space.*

Proof. By Lemma 1.1, we have $(\Phi_M \psi)^{\vee} \mathscr{C} = \psi^{\vee} \Phi_M^{\vee} \mathscr{C} = \psi^{\vee} \Phi_{\tilde{M}}\, \imath$. Using Lemma 1.2, we get that $(\Phi_M \psi)^{\vee} \mathscr{C} = \Phi_M \tilde{\psi} \imath = \epsilon \Phi_M \psi$. Thus $\Phi_M \psi$ is an ϵ-symmetric space. □

We need the following lemma.

LEMMA 1.4. *Let M be a finite-length A-module and $\psi : M \longrightarrow M^{\vee}$ be an ϵ-symmetric space. If (M, ψ) is stably metabolic, then it is metabolic.*

Proof. If M is an A/m-module, then the result follows from the corresponding result for ϵ-symmetric spaces over the field A/m. We reduce the general case to the above case by induction on the length of M. Assume that the length of M is at least 2. Let V be a maximal submodule of M which is an A/m-module. Suppose that ψ restricted to V is singular. Then there exists a non-zero submodule L of V such that

$$L \subset L^{\perp} = \ker(M \xrightarrow{\ i^{\vee}\psi\ } L^{\vee})$$

and ψ induces an ϵ-symmetric form $\bar{\psi}$ on $L^{\perp}/L$ which is Witt equivalent to (M, ψ). Suppose that (M, ψ) is stably metabolic. Then $(L^{\perp}/L, \bar{\psi})$ is stably metabolic. By induction there exists a submodule N_1 of $L^{\perp}/L$ such that

$$0 \longrightarrow N_1 \xrightarrow{\ i\ } L^{\perp}/L \xrightarrow{\ i^{\vee}\bar{\psi}\ } N_1^{\vee} \longrightarrow 0$$

is exact. Let N be the submodule of M containing L such that $N/L = N_1$. Then it is easy to see that the sequence

$$0 \longrightarrow N \xrightarrow{\ i\ } M \xrightarrow{\ i^\vee \psi\ } N^\vee \longrightarrow 0$$

is exact and (M, ψ) is metabolic. We may therefore assume that ψ restricted to V is non-singular. Then $(M, \psi) \simeq (V, \psi|_V) \perp (M_1, \psi_1)$. If $M_1 \neq 0$, then M_1 contains a non-zero submodule which is an A/m-module, contradicting the maximality of V. Thus $M_1 = 0$ and $M = V$ is an A/m-module. This completes the proof of the lemma. □

Let X be a scheme such that 2 is invertible in $\Gamma(X)$. Let $\mathscr{E}$ be a vector bundle over X of finite rank. An *ϵ-symmetric space* on $\mathscr{E}$ is an isomorphism $\mathbf{q}: \mathscr{E} \longrightarrow \mathscr{E}^* = \mathrm{Hom}(\mathscr{E}, \mathscr{O}_X)$ such that $\mathbf{q}^*\mathscr{C} = \epsilon \mathbf{q}$, where $\mathscr{C}: \mathscr{E} \longrightarrow \mathscr{E}^{**}$ is the canonical identification. Let $W^\epsilon(X)$ be the Witt group of ϵ-symmetric spaces on vector bundles over X [**3**, p. 144]. If $X = \mathrm{Spec}(A)$, then we denote $W^\epsilon(X)$ by $W^\epsilon(A)$.

Throughout this paper, by an A-module we mean a finitely generated A-module. We call an ϵ-symmetric space simply a *quadratic space* if $\epsilon = +1$ and a *symplectic space* if $\epsilon = -1$. We also denote $W^{+1}(X)$ by $W(X)$. For a vector bundle $\mathscr{E}$ over X, we denote the hyperbolic space on $\mathscr{E}$ by $\mathbb{H}(\mathscr{E})$ [**3**, p. 130].

2. *Reflexive modules*

Let A be a regular local ring of dimension 3 with 2 invertible. An A-module E is said to be *reflexive* if it is finitely generated and the canonical homomorphism $E \longrightarrow E^{**}$ is an isomorphism. For a reflexive A-module E we use the canonical isomorphism to identify E^{**} with E. It is well known that a reflexive module over a regular ring of dimension 3 has projective dimension at most 1. Let E be a reflexive A-module and $M = \mathrm{Ext}^1(E^*, A)$, where $E^* = \mathrm{Hom}_A(E, A)$. Since reflexive modules over regular rings of dimension at most 2 are projective, M is a finite-length A-module. We define a homomorphism $\beta_E: \mathrm{Ext}^1(E, A) \longrightarrow M^\vee = \mathrm{Ext}^3(M, A)$ as follows. Let

$$0 \longrightarrow P_1 \xrightarrow{\ \partial_1\ } P_0 \xrightarrow{\ \partial_0\ } E^* \longrightarrow 0 \tag{2.1}$$

be a projective resolution of E^*. Then by dualizing, we get an exact sequence

$$0 \longrightarrow E \xrightarrow{\ \partial_0^*\ } P_0^* \xrightarrow{\ \partial_1^*\ } P_1^* \xrightarrow{\ \delta\ } \mathrm{Ext}^1(E^*, A) = M \longrightarrow 0$$

where δ is defined by push-outs. We have the following lemma.

LEMMA 2.1. *The Yoneda composition* [**4**, p. 82] $\beta_E: \mathrm{Ext}^1(E, A) \longrightarrow M^\vee$ *given by*

$$\beta_E(0 \longrightarrow A \xrightarrow{\ \eta\ } Z \xrightarrow{\ \eta'\ } E \longrightarrow 0) = (0 \longrightarrow A \xrightarrow{\ \eta\ } Z \xrightarrow{\ \partial_0^*\eta'\ } P_0^* \xrightarrow{\ \partial_1^*\ } P_1^* \xrightarrow{\ \delta\ } M \longrightarrow 0)$$

is an isomorphism and is independent of the choice of the projective resolution (2.1) *of* E^*.

Proof. Consider the long exact sequence of cohomology associated to the short exact sequences

$$0 \longrightarrow E \xrightarrow{\ \partial_0^*\ } P_0^* \xrightarrow{\ \partial_1^*\ } \ker(\delta) \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \ker(\delta) \hookrightarrow P_1^* \xrightarrow{\ \delta\ } M \longrightarrow 0.$$

Since $\mathrm{Ext}^i(M, A) = 0$ for $i \leqslant 2$ and P_1^* is a projective module, $\mathrm{Ext}^1(\ker(\delta), A) = 0$ and the connecting homomorphisms

$$\mathrm{Ext}^1(E, A) \longrightarrow \mathrm{Ext}^2(\ker(\delta), A) \quad \text{and} \quad \mathrm{Ext}^2(\ker(\delta), A) \longrightarrow \mathrm{Ext}^3(M, A)$$

induced by the above short exact sequences are isomorphisms. Since β_E, up to sign, is the composition of these two connecting homomorphisms [**4**, Theorem 9.1, p. 97], β is an isomorphism.

Suppose that

$$0 \longrightarrow F_1 \xrightarrow{\partial_1'} F_0 \xrightarrow{\partial_0'} E^* \longrightarrow 0$$

is another projective resolution of E^*. Then by lifting the identity map on E^*, we get homomorphisms $P_i \longrightarrow F_i$, $i = 0, 1$, such that Figure 6 is commutative.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & E^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & & \\
0 & \longrightarrow & F_1 & \xrightarrow{\partial_1'} & F_0 & \xrightarrow{\partial_0'} & E^* & \longrightarrow & 0
\end{array}$$

FIGURE 6.

By dualizing this diagram we get a commutative diagram (Figure 7) where δ' is defined by push-outs.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & E & \xrightarrow{\partial_0'^*} & F_0^* & \xrightarrow{\partial_1'^*} & F_1^* & \xrightarrow{\delta'} & M & \longrightarrow & 0 \\
 & & \downarrow \mathrm{id} & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & & \\
0 & \longrightarrow & E & \xrightarrow{\partial_0^*} & P_0^* & \xrightarrow{\partial_1^*} & P_1^* & \xrightarrow{\delta} & M & \longrightarrow & 0
\end{array}$$

FIGURE 7.

This implies that

$$(0 \longrightarrow E \xrightarrow{\partial_0'^*} F_0^* \xrightarrow{\partial_1'^*} F_1^* \xrightarrow{\delta'} M \longrightarrow 0) = (0 \longrightarrow E \xrightarrow{\partial_0^*} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} M \longrightarrow 0)$$

in $\mathrm{Ext}^2(M, E)$. Thus the homomorphism β_E is independent of the choice of the projective resolution of E^*. □

LEMMA 2.2. (i) *For any reflexive A-module E we have*

$$\beta_{E^*} = -\beta_E^{\vee} \mathscr{C}.$$

(ii) *Let E and E′ be reflexive A-modules. Then, for any isomorphism* $f\colon E \longrightarrow E'$, *we have*

$$\mathrm{Ext}^1(f^*)^{\vee} \beta_{E'} = \beta_E \, \mathrm{Ext}^1(f).$$

Proof. Let $0 \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} E^* \longrightarrow 0$ and $0 \longrightarrow F_1 \xrightarrow{\partial_1'} F_0 \xrightarrow{\partial_0'} E \longrightarrow 0$ be projective resolutions of E^* and E respectively. By dualizing these exact sequences, we get exact sequences

$$0 \longrightarrow E \xrightarrow{\partial_0^*} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} \operatorname{Ext}^1(E^*, A) \longrightarrow 0$$

and

$$0 \longrightarrow E^* \xrightarrow{\partial_0'^*} F_0^* \xrightarrow{\partial_1'^*} F_1^* \xrightarrow{\delta_1} \operatorname{Ext}^1(E, A) \longrightarrow 0.$$

Let $\zeta = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\beta} E^* \longrightarrow 0) \in \operatorname{Ext}^1(E^*, A)$. Since P_0 and P_1 are projective, there exist homomorphisms $f: P_1 \longrightarrow A$ and $g: P_0 \longrightarrow Z$ such that Figure 8 is commutative.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & E^* & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow \mathrm{id} & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & Z & \xrightarrow{\beta} & E^* & \longrightarrow & 0
\end{array}$$

FIGURE 8.

By the definition of δ we have $\delta(f) = \zeta$. Since

$$0 \longrightarrow F_1 \xrightarrow{\partial_1'} F_0 \xrightarrow{\partial_0^* \partial_0'} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} \operatorname{Ext}^1(E^*, A) \longrightarrow 0$$

is a projective resolution of $\operatorname{Ext}^1(E^*, A)$, by dualizing it we get an exact sequence

$$0 \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0'^* \partial_0} F_0^* \xrightarrow{\partial_1'^*} F_1^* \xrightarrow{\delta_1'} \operatorname{Ext}^1(E^*, A)^\vee \longrightarrow 0.$$

Thus $\mathscr{C}(\zeta) = -\xi$, where

$$\xi = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\partial_0'^* \beta} F_0^* \xrightarrow{\partial_1'^*} F_1^* \xrightarrow{\delta_1'} \operatorname{Ext}^1(E^*, A)^\vee \longrightarrow 0).$$

From the definitions of δ, δ' and β_E, it follows that Figure 9 is commutative.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & E^* & \xrightarrow{\partial_0'^*} & F_0^* & \xrightarrow{\partial_1'^*} & F_1^* & \xrightarrow{\delta_1} & \operatorname{Ext}^1(E, A) & \longrightarrow & 0 \\
 & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \beta_E & & \\
0 & \longrightarrow & E^* & \xrightarrow{\partial_0'^*} & F_0^* & \xrightarrow{\partial_1'^*} & F_1^* & \xrightarrow{\delta_1'} & \operatorname{Ext}^1(E^*, A)^\vee & \longrightarrow & 0
\end{array}$$

FIGURE 9.

It follows from the definition of $\beta_E^\vee$ that

$$\beta_E^\vee(\xi) = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\partial_0'^* \beta} F_0^* \xrightarrow{\partial_1'^*} F_1^* \xrightarrow{\delta_1} \operatorname{Ext}^1(E, A) \longrightarrow 0).$$

On the other hand, we have

$$\beta_{E^*}(\zeta) = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\partial_0'^* \beta} F_0^* \xrightarrow{\partial_1'^*} F_1^* \xrightarrow{\delta_1} \operatorname{Ext}^1(E, A) \longrightarrow 0) = \beta_E^\vee(\xi).$$

Thus $-\beta_E^\vee \mathscr{C} = \beta_{E^*}$.

Let $f\colon E \longrightarrow E'$ be an isomorphism. Then $0 \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{f^{*-1}\partial_0} E'^* \longrightarrow 0$ is a projective resolution of E'^*. By dualizing it we get an exact sequence

$$0 \longrightarrow E' \xrightarrow{\partial_0^* f^{-1}} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta_2} \operatorname{Ext}^1(E'^*, A) \longrightarrow 0$$

with $\delta_2 = \operatorname{Ext}^1(f^*)\ \delta$. Let $\zeta = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\beta} E' \longrightarrow 0) \in \operatorname{Ext}^1(E', A)$. Then

$$\beta_{E'}(\zeta) = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\partial_0^* f^{-1}\beta} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta_2} \operatorname{Ext}^1(E'^*, A) \longrightarrow 0)$$

and

$$\operatorname{Ext}^1(f^*)^\vee \beta_{E'}(\zeta) = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\partial_0^* f^{-1}\beta} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} \operatorname{Ext}^1(E^*, A) \longrightarrow 0)$$

since $\delta = \operatorname{Ext}^1(f^*)^{-1}\delta_2$. On the other hand, we have

$$\operatorname{Ext}^1(f)(\zeta) = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{f^{-1}\beta} E \longrightarrow 0)$$

and

$$\begin{aligned}\beta_E \operatorname{Ext}^1(f)(\zeta) &= (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\partial_0^* f^{-1}\beta} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} \operatorname{Ext}^1(E^*, A) \longrightarrow 0)\\ &= \operatorname{Ext}^1(f^*)^\vee \beta_{E'}(\zeta).\end{aligned}$$

This proves the lemma. □

Let A be any local ring in which 2 is invertible and let m be its maximal ideal. Let E be a reflexive A-module. By an *ϵ-symmetric space* on E we mean an isomorphism $q\colon E \longrightarrow E^*$ such that $q^*\mathscr{C} = \epsilon q$, where $\mathscr{C}\colon E \longrightarrow E^{**}$ is the canonical isomorphism.

Let V be an A-module. By a *unimodular* element of V we mean an element $x \in V$ such that $f(x) = 1$ for some A-linear map $f\colon V \longrightarrow A$. For example, an element $(a_1, \ldots, a_n) \in A^n$ is unimodular if and only if $a_i \notin m$ for some i. Thus, if an A-module V has no unimodular elements and $\eta\colon V \longrightarrow A^n$ is an A-linear map, then $\eta(V) \subset mA^n$.

LEMMA 2.3. *Let E be a reflexive A-module and q be an ϵ-symmetric space on E. Suppose that $E = E_0 \oplus A^n$ with E_0 having no unimodular elements. Then there exist ϵ-symmetric spaces q_1 and q_2 over E_0 and A^n respectively such that*

$$(E, q) \simeq (E_0, q_1) \perp (A^n, q_2).$$

Proof. Let $E = E_0 \oplus A^n$ be such that E_0 has no unimodular elements. Then

$$q = \begin{pmatrix} q_1 & \eta \\ \epsilon\eta^* & q_1' \end{pmatrix}$$

for some $q_1\colon E_0 \longrightarrow E_0^*$, $q_1'\colon A^n \longrightarrow A^{n*}$ and $\eta\colon A^n \longrightarrow E_0^*$. Since E_0 has no unimodular elements, $\eta^*(E_0) \subset mA^{n*}$ and hence $\eta(A^n) \subset mE_0^*$. This implies that

$$q \equiv \begin{pmatrix} q_1 & 0 \\ 0 & q_1' \end{pmatrix} \operatorname{mod} mE^*.$$

Since q is an isomorphism, q_1 and q_1' are isomorphisms. We have

$$\begin{pmatrix} 1 & 0 \\ -\epsilon\eta^* q_1^{-1} & 1 \end{pmatrix}\begin{pmatrix} q_1 & \eta \\ \epsilon\eta^* & q_1' \end{pmatrix}\begin{pmatrix} 1 & -q_1^{-1}\eta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q_1 & 0 \\ 0 & -\eta^* q_1^{-1}\eta + q_1' \end{pmatrix}.$$

Let $q_2 = -\eta^* q_1^{-1}\eta + q_1' : A^n \longrightarrow A^{n*}$. Since $q_1^* = \epsilon q_1$, $(E, q) \simeq (E_0, q_1) \perp (A^n, q_2)$. □

3. *Spaces over the punctured spectrum and on finite-length modules*

We begin by recalling from a paper of Horrocks [**2**] an equivalence between the categories $\Phi\mathscr{P}$ of Φ-equivalence classes of vector bundles on the punctured spectrum of a regular local ring A of dimension 3 and the category $\mathscr{M}$ of finite-length A-modules. Let m be the maximal ideal of A and $Y = \operatorname{Spec}(A)\backslash\{m\}$. Let $\mathscr{E}$ be a vector bundle over Y and $E = \Gamma(\mathscr{E})$ be the module of sections of $\mathscr{E}$. Then E is a reflexive A-module [**2**, Theorem 4.1] and $M = \operatorname{Ext}^1(E^*, A)$, which is isomorphic to $H^1(Y, \mathscr{E})$ [**2**, § 5], is a finite-length A-module [**2**, Corollary 7.2.5]. The functor $T : \Phi\mathscr{P} \longrightarrow \mathscr{M}$ given by $T(\mathscr{E}) = \operatorname{Ext}^1(E^*, A)$ is an equivalence of categories [**2**, Corollary 7.2.5]. Let M be a finite-length A-module. The construction below gives a vector bundle $\mathscr{E}$ on Y such that $T(\mathscr{E}) = M$. Let, in fact,

$$0 \longrightarrow P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\eta} M \longrightarrow 0$$

be a projective resolution of M. Let $E = \ker(\partial_1)$. Then E is an A-module of projective dimension at most 1 and $\operatorname{Ext}^1(E^*, A) = M$. Since M is a finite-length module, for any prime ideal p of A, $p \neq m$, $M_p = 0$ and hence E_p is free. Thus $E = \Gamma(\mathscr{E})$ for some vector bundle $\mathscr{E}$ on Y [**2**, Theorem 4.1].

Let A be a regular local ring of dimension 3 in which 2 is invertible. Let $\mathscr{E}$ be a vector bundle over Y and $\mathbf{q}$ be an ϵ-symmetric space on $\mathscr{E}$. We associate to $(\mathscr{E}, \mathbf{q})$ a $(-\epsilon)$-symmetric space $\rho(\mathbf{q})$ of finite length. The ϵ-symmetric space $\mathbf{q}$ on $\mathscr{E}$ gives rise to an ϵ-symmetric space (E, q), where $E = \Gamma(\mathscr{E})$. Then $M = \operatorname{Ext}^1(E^*, A)$ is a finite-length A-module. The isomorphism $q : E \longrightarrow E^*$ induces an isomorphism $\operatorname{Ext}^1(q) : M = \operatorname{Ext}^1(E^*, A) \longrightarrow \operatorname{Ext}^1(E, A)$. Let $\rho(\mathbf{q}) = \beta_E \operatorname{Ext}^1(q)$. We have the following lemma.

LEMMA 3.1. *$\rho(\mathbf{q}) : M \longrightarrow M^\vee$ is a $(-\epsilon)$-symmetric space.*

Proof. In Figure 10, clearly, all the squares except perhaps the top left one commute.

$$\begin{array}{ccccc}
M = \operatorname{Ext}^1(E^*, A) & \xrightarrow{\operatorname{Ext}^1(q)} & \operatorname{Ext}^1(E, A) & \xrightarrow{\beta_E} & \operatorname{Ext}^1(E^*, A)^\vee = M^\vee \\
\downarrow{\scriptstyle \beta_{E^*}} & & \downarrow{\scriptstyle \beta_E} & & \downarrow{\scriptstyle \mathrm{id}} \\
\operatorname{Ext}^1(E, A)^\vee & \xrightarrow{\epsilon \operatorname{Ext}^1(q)^\vee} & \operatorname{Ext}^1(E^*, A)^\vee & \xrightarrow{\mathrm{id}} & \operatorname{Ext}^1(E^*, A)^\vee \\
\downarrow{\scriptstyle \beta_E^{\vee -1}} & & \downarrow{\scriptstyle \epsilon \operatorname{Ext}^1(q)^{\vee -1}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\operatorname{Ext}^1(E^*, A)^{\vee\vee} & \xrightarrow{\beta_E^\vee} & \operatorname{Ext}^1(E, A)^\vee & \xrightarrow{\epsilon \operatorname{Ext}^1(q)^\vee} & \operatorname{Ext}^1(E^*, A)^\vee
\end{array}$$

FIGURE 10.

Since $q^* = \epsilon q$, by Lemma 2.2 this square also commutes. By Lemma 2.2, the composition of maps on the first column is equal to $-\mathscr{C}$. Thus

$$\rho(q)^\vee \mathscr{C} = \mathrm{Ext}^1(q)^\vee \beta_E^\vee \mathscr{C} = -\epsilon\beta_E \,\mathrm{Ext}^1(q) = -\epsilon\rho(q). \qquad \square$$

LEMMA 3.2. *Let M be a finite-length A-module and ψ be an ϵ-symmetric form on M. Suppose that there exists an exact sequence*

$$N \xrightarrow{\ f\ } M \xrightarrow{\ f^\vee \psi\ } N^\vee$$

of finite-length A-modules. Then (M, ψ) is metabolic.

Proof. Since the map f factors as $N \longrightarrow N/\ker(f) \xrightarrow{\ \bar{f}\ } M$, we have an exact sequence

$$0 \longrightarrow N/\ker(f) \xrightarrow{\ \bar{f}\ } M \xrightarrow{\ \bar{f}^\vee \psi\ } (N/\ker(f))^\vee.$$

Since, the dimension of A being 3, $\mathrm{Ext}^4(L, A) = 0$, the map $\bar{f}^\vee \psi$ is surjective and hence (M, ψ) is metabolic. $\square$

LEMMA 3.3. If $(\mathscr{E}, \mathbf{q})$ *is metabolic, then* $(M, \rho(\mathbf{q}))$ *is metabolic.*

Proof. Suppose that $(\mathscr{E}, \mathbf{q})$ is metabolic. Let $\mathscr{F}$ be a subbundle of $\mathscr{E}$ such that the sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\ i\ } \mathscr{E} \xrightarrow{\ i^*\mathbf{q}\ } \mathscr{F}^* \longrightarrow 0$$

is exact, where $\mathscr{F} \xrightarrow{\ i\ } \mathscr{E}$ is the inclusion. By taking global sections and then applying the Ext functor to the following exact sequence of bundles

$$0 \longrightarrow \mathscr{F} \xrightarrow{\ \mathbf{q}i\ } \mathscr{E}^* \xrightarrow{\ i^*\ } \mathscr{F}^* \longrightarrow 0$$

we get an exact sequence

$$\mathrm{Ext}^1(F^*, A) \longrightarrow \mathrm{Ext}^1(E^*, A) \longrightarrow \mathrm{Ext}^1(F, A)$$

of finite-length modules, where $F = \Gamma(\mathscr{F})$. Let $N = \mathrm{Ext}^1(F^*, A)$. Then the canonical identification of $\mathrm{Ext}^1(F, A)$ with $N^\vee$ gives an exact sequence

$$N \xrightarrow{\ f\ } M \xrightarrow{\ f^\vee \rho(\mathbf{q})\ } N^\vee.$$

Now the lemma follows from Lemma 3.2. $\square$

LEMMA 3.4. *The assignment* $(\mathscr{E}, \mathbf{q}) \mapsto (M, \rho(\mathbf{q}))$ *induces a homomorphism*

$$\rho : W^\epsilon(Y) \longrightarrow W_{\mathrm{fl}}^{-\epsilon}(A).$$

Proof. Since ρ is clearly additive, it is enough to show that ρ takes stably metabolic spaces to metabolic spaces. Let $(\mathscr{E}, \mathbf{q})$ be an ϵ-symmetric space over Y which is stably metabolic. Then there exists a metabolic space $(\mathscr{E}_1, \mathbf{q}_1)$ such that $(\mathscr{E}, \mathbf{q}) \perp (\mathscr{E}_1, \mathbf{q}_1)$ is metabolic. By Lemma 3.3, $\rho(\mathbf{q}_1)$ and $\rho(\mathbf{q} \perp \mathbf{q}_1) = \rho(\mathbf{q}) \perp \rho(\mathbf{q}_1)$ are metabolic. Thus $\rho(\mathbf{q})$ is stably metabolic. $\square$

We note that if $\mathscr{E}$ is a trivial bundle then $M = 0$. Thus, if $(\mathscr{E}, \mathbf{q})$ comes from an ϵ-symmetric space on A, then $\rho(\mathbf{q}) = 0$.

The proof of the following lemma is by straightforward verification; hence we omit it.

LEMMA 3.5. *Let R be a ring. Let $0 \longrightarrow N \xrightarrow{i} M \xrightarrow{j} N' \longrightarrow 0$ be an exact sequence of R-modules. Assume that the projective dimensions of N and N' are finite. Let*

$$0 \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\alpha} N \longrightarrow 0$$

and

$$0 \longrightarrow Q_n \xrightarrow{\partial'_n} Q_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \longrightarrow Q_1 \xrightarrow{\partial'_1} Q_0 \xrightarrow{\beta} N' \longrightarrow 0$$

be projective resolutions of N and N' respectively. Let, for $l \geqslant 1$, $\phi_l : Q_l \longrightarrow P_{l-1}$ and $\theta : Q_0 \longrightarrow M$ be R-linear homomorphisms. Let

$$\delta_l = \begin{pmatrix} \partial_l & (-1)^l \phi_l \\ 0 & \partial'_l \end{pmatrix}.$$

Then Figure 11 is commutative if and only if Figure 12 is commutative.

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & P_n & \xrightarrow{\partial_n} & P_{n-1} & \cdots\longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\alpha} & N & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & P_n\oplus Q_n & \xrightarrow{\delta_n} & P_{n-1}\oplus Q_{n-1} & \cdots\longrightarrow & P_1\oplus Q_1 & \xrightarrow{\delta_1} & P_0\oplus Q_0 & \xrightarrow{(i\alpha,\,\theta)} & M & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & Q_n & \xrightarrow{\partial'_n} & Q_{n-1} & \cdots\longrightarrow & Q_1 & \xrightarrow{\partial'_1} & Q_0 & \xrightarrow{\beta} & N' & \longrightarrow & 0
\end{array}$$

FIGURE 11.

$$\begin{array}{ccccccccccccccccc}
 & & 0 & \longrightarrow & Q_n & \xrightarrow{\partial'_n} & Q_{n-1} & \xrightarrow{\partial'_{n-1}} \cdots \longrightarrow & Q_2 & \xrightarrow{\partial'_2} & Q_1 & \xrightarrow{\partial'_1} & Q_0 & \xrightarrow{\beta} & N' & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & P_n & \xrightarrow{\partial_n} & P_{n-1} & \xrightarrow{\partial_{n-1}} & P_{n-2} & \xrightarrow{\partial_{n-2}} \cdots \longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{i\alpha} & M & \xrightarrow{j} & N' & \longrightarrow & 0
\end{array}$$

FIGURE 12.

PROPOSITION 3.6. *Let $(\mathscr{E}, \mathbf{q})$ be an ϵ-symmetric space over Y. Suppose that $E = \Gamma(\mathscr{E})$ has no unimodular elements and that $\rho(\mathbf{q})$ is metabolic. Then there exist ϵ-symmetric spaces $\mathbf{q}_1$ and $\mathbf{q}_2$ on $\mathscr{E}$ and $\mathcal{O}_Y^n$ respectively such that $\mathbf{q}_1 \perp \mathbf{q}_2$ is metabolic and $\rho(\mathbf{q}) \simeq \rho(\mathbf{q}_1)$.*

Proof. Let $M = \mathrm{Ext}^1(E^*, A)$ and $\rho(\mathbf{q})$ be the ϵ-symmetric space on M. Since $\rho(\mathbf{q})$ is metabolic, there exists an exact sequence

$$0 \longrightarrow N \xrightarrow{\ i\ } M \xrightarrow{i^\vee \rho(q)} N^\vee \longrightarrow 0.$$

Let

$$0 \longrightarrow Q_3 \xrightarrow{\partial_3} Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\eta} N \longrightarrow 0$$

be a projective resolution of N. By dualizing this resolution, we get a projective resolution

$$0 \longrightarrow Q_0^* \xrightarrow{\partial_1^*} Q_1^* \xrightarrow{\partial_2^*} Q_2^* \xrightarrow{\partial_3^*} Q_3^* \xrightarrow{\eta'} N^\vee \longrightarrow 0$$

of $N^\vee$. By lifting the identity map of $N^\vee$, we obtain a commutative diagram (Figure 13).

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^\vee & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{i\eta} & M & \xrightarrow{i^\vee \rho(q)} & N^\vee & \longrightarrow & 0
\end{array}$$

FIGURE 13.

Let

$$\delta_1 = \begin{pmatrix} \partial_1 & -\phi_1 \\ 0 & \partial_3^* \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} \partial_2 & \phi_2 \\ 0 & \partial_2^* \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} \partial_3 & -\phi_3 \\ 0 & \partial_1^* \end{pmatrix}.$$

By Lemma 3.5, Figure 14 is commutative.

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\eta} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & Q_3 \oplus Q_0^* & \xrightarrow{\delta_3} & Q_2 \oplus Q_1^* & \xrightarrow{\delta_2} & Q_1 \oplus Q_2^* & \xrightarrow{\delta_1} & Q_0 \oplus Q_3^* & \xrightarrow{(i\eta, \theta)} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i^\vee \rho(q)} & & \\
0 & \longrightarrow & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^\vee & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

FIGURE 14.

Since the first row, the last rows and all the columns in Figure 14 are exact and the diagram is commutative, from the long exact homology sequence [**4**, Theorem 4.1, p. 45] we get that the middle row is also exact. By dualizing Figure 14, we get the commutative diagram in Figure 15, with exact rows and columns.

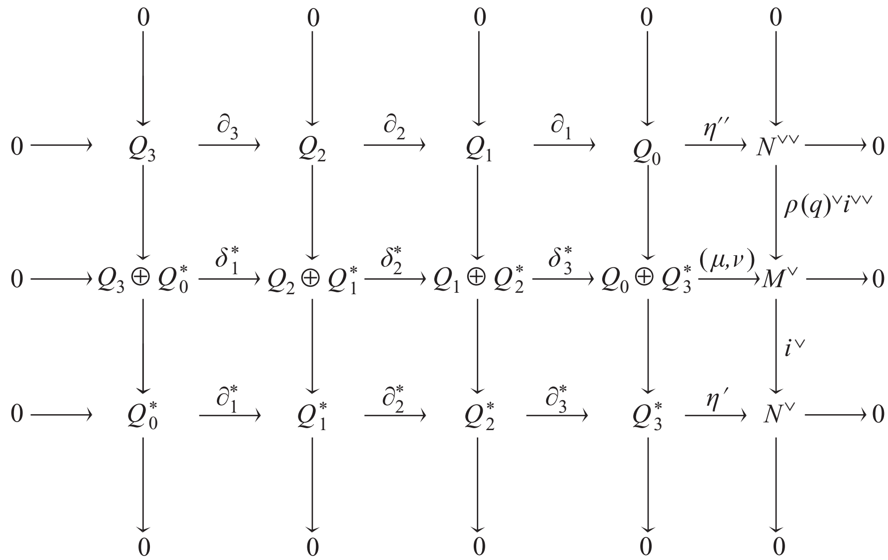

FIGURE 15.

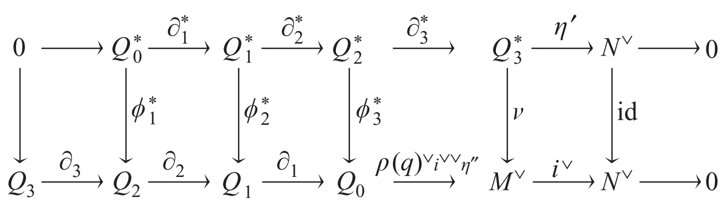

FIGURE 16.

By Lemma 3.5, Figure 16 is commutative. From the definition of η'' and $\mathscr{C}: N \longrightarrow N^{\vee\vee}$ (cf. Figure 2) it follows that $\mathscr{C}\eta = -\eta''$. Since $\rho(q)^{\vee}\mathscr{C} = -\epsilon\rho(q)$, we have the commutative diagram in Figure 17, where $\nu' = \epsilon\rho(q)^{-1}\nu$.

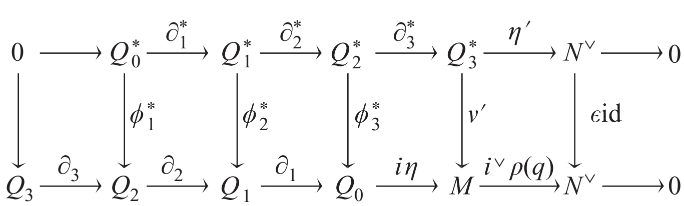

FIGURE 17.

From Figure 13 and Figure 17 we get maps $s_1 : Q_2^* \longrightarrow Q_1$ and $s_2 : Q_1^* \longrightarrow Q_2$ such that $\phi_2 - \epsilon\phi_2^* = \partial_2 s_2 - s_1 \partial_2^*$. Let $\phi = \phi_2 - \epsilon\partial_2 s_1^*$. Then we have

$$\begin{aligned}\partial_1 \phi &= \partial_1 \phi_2 \\ &= \partial_1(\phi_2 - \epsilon\phi_2^*) + \epsilon\partial_1\phi_2^* \\ &= \partial_1(\partial_2 s_2 - s_1 \partial_2^*) + \epsilon\partial_1 \phi_2^* \\ &= \epsilon\partial_1(\phi_2^* - \epsilon s_1 \partial_2^*) \\ &= \epsilon\partial_1 \phi^*.\end{aligned}$$

Let

$$\delta = \begin{pmatrix} \partial_2 & \dfrac{\phi + \epsilon\phi^*}{2} \\ 0 & \epsilon\partial_2^* \end{pmatrix}.$$

It is easy to see that Figure 18 commutes.

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\eta} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & Q_3 \oplus Q_0^* & \xrightarrow{\delta_1^*} & Q_2 \oplus Q_1^* & \xrightarrow{\delta} & Q_1 \oplus Q_2^* & \xrightarrow{\delta_1} & Q_0 \oplus Q_3^* & \xrightarrow{(\eta,\theta)} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i^\vee \rho(q)} & & \\
0 & \longrightarrow & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\epsilon\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^\vee & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

FIGURE 18.

Since the first row, the last row and all the columns are exact, the middle row is also exact. Let $E' = \ker(\delta_1)$. Since $\delta^*\mathscr{C} = \epsilon\delta$, from the middle row of Figure 18 it is easy to see that δ induces an ϵ-symmetric isomorphism $q' : E' \longrightarrow E'^*$. Let $(\mathscr{E}', \mathbf{q}')$ be the ϵ-symmetric space over Y with $\Gamma(\mathscr{E}') = E'$ and $\Gamma(\mathbf{q}') = q'$. Since $\mathrm{Ext}^1(E', A) \simeq M = \mathrm{Ext}^1(E^*, A)$ and E has no unimodular elements, by [**2**, Corollary 7.2.5, Lemma 7.1] we have $E' = E \oplus A^n$. Then by Lemma 2.3, $(E', q') \simeq (E, q_1) \perp (A^n, q_2)$ for some ϵ-symmetric spaces q_1 and q_2 on E and A^n respectively. Let $\mathbf{q}_1$ be the ϵ-symmetric space on $\mathscr{E}$ such that $\Gamma(\mathbf{q}_1) = q_1$. Let $F = \ker(\partial_1)$ and $\mathscr{F}$ be the vector bundle over Y with $\Gamma(\mathscr{F}) = F$. Then using Figure 18, it is easy to see that $\mathscr{F}$ is a Lagrangian for $(\mathscr{E}', \mathbf{q}') \simeq (\mathscr{E}, \mathbf{q}_1) \perp (\mathscr{O}_Y^n, \mathbf{q}_2)$, where $\Gamma(\mathbf{q}_2) = q_2$. Since the map $E' \longrightarrow F^*$ induced by Figure 18 induces $i^\vee \rho(\mathbf{q}) : M \longrightarrow N^\vee$, we have $i^\vee \rho(\mathbf{q}) = i^\vee \rho(\mathbf{q}_1)$. Thus, by Lemma 3.7 below, we have $\rho(\mathbf{q}) \simeq \rho(\mathbf{q}_1)$. □

LEMMA 3.7. *Let ψ_1 and ψ_2 be two ϵ-symmetric spaces on M. Suppose there exists a submodule N such that*

$$0 \longrightarrow N \xrightarrow{\;i\;} M \xrightarrow{\;i^\vee \psi_1\;} N^\vee \longrightarrow 0$$

is exact and $i^\vee \psi_1 = i^\vee \psi_2$. Then $\psi_1 \simeq \psi_2$.

Proof. Since $i^\vee \psi_1 = i^\vee \psi_2$, there exists a map $\theta : M \longrightarrow N$ such that $\psi_1^{-1}\psi_2 - 1 = i\theta$, that is, $\psi_1 i\theta = \psi_2 - \psi_1 = \theta^\vee i^\vee \psi_1$. We have

$$\begin{aligned}
\frac{(1+\theta^\vee i^\vee)}{2}\psi_1\frac{(1+i\theta)}{2} &= \frac{(\psi_1+\theta^\vee i^\vee \psi_1)}{2}\frac{(1+i\theta)}{2} \\
&= \psi_1 + \psi_1\frac{i\theta}{2} + \frac{\theta^\vee i^\vee}{2}\psi_1 + \frac{\theta^\vee i^\vee}{2}\psi_1\frac{i\theta}{2} \\
&= \psi_1 + \frac{\psi_2-\psi_1}{2} + \frac{\psi_2-\psi_1}{2} \\
&= \psi_2.
\end{aligned}$$

□

4. *The Witt groups of the punctured spectrum and purity*

Let A be a regular local ring of dimension 3 with 2 invertible. Let $Y = \mathrm{Spec}(A)\backslash\{m\}$, where m is the maximal ideal of A.

PROPOSITION 4.1. *Let $\mathscr{E}$ be a vector bundle on Y and $\mathbf{q} : \mathscr{E} \longrightarrow \mathscr{E}^*$ be an ϵ-symmetric isomorphism. Suppose that $\Gamma(\mathscr{E})$ has no unimodular elements. If $\rho(\mathbf{q})$ is isomorphic to a hyperbolic space, then $\mathbf{q}$ is in the image of $W^\epsilon(A)$.*

Proof. Let N be a finite-length A-module such that $(M, \rho(\mathbf{q}))$ is isomorphic to the hyperbolic space $\mathbb{H}(N)$. Let $\mathscr{F}$ be the vector bundle on Y with $\Gamma(\mathscr{F})$ having no unimodular elements and such that $H^1(Y, \mathscr{F}) \simeq N$ (cf. §3). Since

$$H^1(Y, \mathscr{E}) \simeq N \oplus N^\vee \simeq H^1(Y, \mathscr{F} \oplus \mathscr{F}^*)$$

with $\Gamma(\mathscr{E})$ and $\Gamma(\mathscr{F} \oplus \mathscr{F}^*)$ admitting no unimodular elements, by [**2**, Lemma 7.1, Corollary 7.2.5] we can and do identify $\mathscr{E}$ with $\mathscr{F} \oplus \mathscr{F}^*$. Let $\tilde{\psi}$ be an isometry of $\rho(\mathbf{q})$ with

$$\rho\left(\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}\right).$$

Then by [**2**, Corollary 7.2.5] there exists an automorphism ψ of $\mathscr{E}$ such that $H^1(\psi) = \tilde{\psi}$. By the definition of ρ we have

$$\begin{aligned}
\rho(\psi^*\mathbf{q}\psi) &= \beta_E \,\mathrm{Ext}^1(\Gamma(\psi^*) q \Gamma(\psi)) \\
&= \beta_E \,\mathrm{Ext}^1(\Gamma(\psi^*))\,\mathrm{Ext}^1(q)\,\mathrm{Ext}^1(\Gamma(\psi))
\end{aligned}$$

where $E = \Gamma(\mathscr{E})$ and $q = \Gamma(\mathbf{q})$. By Lemma 2.2(ii), we have $\beta_E \,\mathrm{Ext}^1(\Gamma(\psi^*)) = \mathrm{Ext}^1(\Gamma(\psi))^\vee \beta_E$, so that

$$\begin{aligned}
\rho(\psi^*\mathbf{q}\psi) &= \mathrm{Ext}^1(\Gamma(\psi))^\vee \beta_E \,\mathrm{Ext}^1(q)\,\mathrm{Ext}^1(\Gamma(\psi)) \\
&= \mathrm{Ext}^1(\Gamma(\psi))^\vee \rho(\mathbf{q})\,\mathrm{Ext}^1(\Gamma(\psi)) \\
&= H^1(\psi)^\vee \rho(\mathbf{q}) H^1(\psi) \\
&= \tilde{\psi}\rho(\mathbf{q})\tilde{\psi}.
\end{aligned}$$

We replace $\mathbf{q}$ by $\psi^*\mathbf{q}\psi$ and assume that

$$\rho(\mathbf{q}) = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}.$$

Let

$$\mathbf{q} = \begin{pmatrix} \alpha & \delta \\ \epsilon\delta^* & \beta \end{pmatrix}$$

with $\alpha : \mathscr{F} \longrightarrow \mathscr{F}^*$, $\beta : \mathscr{F}^* \longrightarrow \mathscr{F}$, $\delta : \mathscr{F}^* \longrightarrow \mathscr{F}^*$ maps such that $\alpha^* = \epsilon\alpha$, $\beta^* = \epsilon\beta$. Since

$$\rho(\mathbf{q}) = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix},$$

$H^1(\alpha) = 0$ and $H^1(\beta) = 0$. Therefore, by [**2**, Corollary 7.2.5], there exist

$$f_1 : \mathscr{F} \longrightarrow \mathscr{O}_Y^n, \quad f_2 : \mathscr{F}^* \longrightarrow \mathscr{O}_Y^n, \quad g_1 : \mathscr{O}_Y^n \longrightarrow \mathscr{F}^* \quad \text{and} \quad g_2 : \mathscr{O}_Y^n \longrightarrow \mathscr{F}$$

such that $\alpha = g_1 f_1$ and $\beta = g_2 f_2$. Let us consider the automorphism

$$\psi = \begin{pmatrix} 1 & 0 & -g_1/2 & f_1^* \\ 0 & 1 & -g_2/2 & f_2^* \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

of $\mathscr{F} \oplus \mathscr{F}^* \oplus \mathscr{O}_Y^n \oplus \mathscr{O}_Y^{n*}$. We have

$$\mathbf{q}' = \psi \begin{pmatrix} \alpha & \delta & 0 & 0 \\ \epsilon\delta^* & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \epsilon & 0 \end{pmatrix} \qquad \psi^* = \begin{pmatrix} 0 & X & \epsilon f_1^* & -g_1/2 \\ \epsilon X^* & 0 & \epsilon f_2^* & -g_2/2 \\ f_1 & f_2 & 0 & 1 \\ -\epsilon g_1^*/2 & -\epsilon g_2^*/2 & \epsilon & 0 \end{pmatrix}$$

where $X = \delta - \epsilon f_1^* g_2^*/2 - g_1 f_2/2$. Since $\Gamma(\mathscr{F})$ and $\Gamma(\mathscr{F}^*)$ admit no unimodular elements, $f_1 \equiv f_2 \equiv 0 \bmod m$. Hence X is an isomorphism. Since $\mathbf{q}'$ restricted to $\mathscr{F} \oplus \mathscr{F}^*$ is

$$\begin{pmatrix} 0 & X \\ \epsilon X^* & 0 \end{pmatrix}$$

with X an isomorphism, $\mathbf{q}'$ splits as

$$\begin{pmatrix} 0 & X \\ \epsilon X^* & 0 \end{pmatrix} \perp \mathbf{q}''$$

for some $\mathbf{q}''$ supported on a bundle $\mathscr{E}''$. Since $\mathscr{E} \oplus \mathscr{O}_Y^n \oplus \mathscr{O}_Y^{n^*} \simeq \mathscr{E} \oplus \mathscr{E}''$, by [**2**, Corollary 7.2.5], $\mathscr{E}''$ is a trivial bundle and hence $\mathbf{q}''$ is in the image of $W^\epsilon(A)$. Since

$$\mathbf{q} \perp H(\mathscr{O}_Y^n) \simeq \mathbf{q}' \simeq \begin{pmatrix} 0 & X \\ \epsilon X^* & 0 \end{pmatrix} \perp \mathbf{q}''$$

it follows that $\mathbf{q}$ is in the image of $W^\epsilon(A)$. □

LEMMA 4.2. *Let M be a finite-length A-module and $\psi : M \longrightarrow M^\vee$ be an ϵ-symmetric isomorphism. Then there exists a vector bundle $\mathscr{E}$ over Y with a $(-\epsilon)$-symmetric isomorphism $\mathbf{q} : \mathscr{E} \longrightarrow \mathscr{E}^*$ such that $\rho(\mathbf{q}) = \psi$.*

Proof. Let $\mathscr{E}$ be a vector bundle on Y such that $H^1(Y, \mathscr{E}) = M$ (cf. §3) and $E = \Gamma(\mathscr{E})$ has no unimodular elements. Then (cf. §3) the projective dimension of E is less than or equal to 1 and $\mathrm{Ext}^1(E^*, A) \simeq M$. Let

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E \longrightarrow 0$$

and

$$0 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow E^* \longrightarrow 0$$

be projective resolutions of E and E^* respectively. By dualizing the projective resolution of E^* we get an exact sequence

$$0 \longrightarrow E \longrightarrow Q_0^* \longrightarrow Q_1^* \longrightarrow M \longrightarrow 0.$$

By taking the Yoneda composition of this exact sequence with the projective resolution of E we get a projective resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Q_0^* \longrightarrow Q_1^* \longrightarrow M \longrightarrow 0$$

of M. By dualizing this we get a projective resolution

$$0 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow M^\vee \longrightarrow 0$$

of $M^\vee$. By lifting the ϵ-symmetric isomorphism $\psi : M \longrightarrow M^\vee$, we get a commutative diagram of exact sequences (Figure 19).

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & Q_0^* & \longrightarrow & Q_1^* & \longrightarrow & M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle 3} & & \Big\downarrow{\scriptstyle 2} & & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow & & \\
0 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & P_0^* & \longrightarrow & P_1^* & \longrightarrow & M^\vee & \longrightarrow & 0
\end{array}$$

FIGURE 19.

Since E has no unimodular elements it is easy to see, as in the proof of Proposition 3.6, that Figure 19 induces a $(-\epsilon)$-symmetric space $\mathbf{q}$ on $\mathscr{E}$ such that $\rho(\mathbf{q}) = \psi$. □

THEOREM 4.3. *Let A be a regular local ring of dimension 3 and let m be its maximal ideal. Assume that 2 is invertible in A. Let $Y = \mathrm{Spec}(A) \backslash \{m\}$. Then the complex*

$$0 \longrightarrow W^\epsilon(A) \xrightarrow{\ \iota\ } W^\epsilon(Y) \xrightarrow{\ \rho\ } W_{\mathrm{fl}}^{-\epsilon}(A) \longrightarrow 0$$

is exact, where ι is induced by the restriction.

Proof. If $\epsilon = 1$, then the injectivity of ι follows from the injectivity of the canonical homomorphism $W(A) \longrightarrow W(K)$ [**6**, Theorem 23], where K is the quotient field of A. If $\epsilon = -1$, ι is injective because $W^{-1}(A) = 0$.

We now prove the exactness in the middle. As we remarked in §3, $\rho\iota = 0$. Let $(\mathscr{E}, \mathbf{q})$ be an ϵ-symmetric space over Y such that $\rho(\mathbf{q})$ is zero in $W_{\mathrm{fl}}^{-\epsilon}(A)$. Then, by Lemma 1.4, $\rho(\mathbf{q})$ is metabolic. We show that $(\mathscr{E}, \mathbf{q})$ is in the image of ι. In view of Lemma 2.3, we assume that $\Gamma(\mathscr{E})$ has no unimodular elements. Then, by Proposition 3.6, there exist ϵ-symmetric spaces $\mathbf{q}_1$ and $\mathbf{q}_2$ supported respectively on $\mathscr{E}$ and $\mathscr{O}_Y^n$ for some integer n, such that $\mathbf{q}_1 \perp \mathbf{q}_2$ is metabolic and $\rho(\mathbf{q}) \simeq \rho(\mathbf{q}_1)$. Thus $\rho(\mathbf{q} \perp -\mathbf{q}_1)$ is isomorphic to a hyperbolic space. Since $\Gamma(\mathscr{E} \oplus \mathscr{E})$ has no unimodular elements, it follows from Proposition 4.1 that $\mathbf{q}\perp -\mathbf{q}_1$ is in the image of ι. Since $\mathbf{q}_2$ is in the image of ι and $\mathbf{q}_1\perp\mathbf{q}_2$ is metabolic, $\mathbf{q}_1$ and hence $\mathbf{q}$ is in the image of ι.

The surjectivity of ρ follows from Lemma 4.2. □

Let A be any regular ring. Let $\mathrm{Spec}^1(A)$ denote the set of prime ideals of A of height 1. Then for any $P\in\mathrm{Spec}^1(A)$, the local ring A_P is a discrete valuation ring. Let $\partial_P\colon W(K)\longrightarrow W(A_P/PA_P)$ denote the second residue homomorphism with respect to some choice of a parameter of PA_P, where K is the quotient field of A.

Corollary 4.4. *Let A be a regular local ring of dimension 3, m be its maximal ideal and K be its quotient field. Assume that 2 is invertible in A. The sequence*

$$0\longrightarrow W(A)\longrightarrow W(K)\xrightarrow{\oplus\partial_P}\bigoplus_{P\in\mathrm{Spec}^1(A)} W(A_P/PA_P)$$

is exact.

Proof. The injectivity of $W(A)\longrightarrow W(K)$ is proved in [**6**, Theorem 23]. Since $W^{-1}_{\mathrm{fl}}(A)\simeq W^{-1}(A/m)=0$, by Theorem 4.3 we have $W(A)\simeq W(Y)$. Thus it is enough to prove that the complex

$$W(Y)\longrightarrow W(K)\xrightarrow{\oplus\partial_P}\bigoplus_{P\in\mathrm{Spec}^1(A)} W(A_P/PA_P)$$

is exact. Let q be a quadratic space over K such that $\partial_P(q)=0$ for all height 1 prime ideals P of A. Since Y is a regular scheme of dimension 2, by [**1**, 2.5, p. 112], there exists a quadratic space $(\mathscr{E},\mathbf{q})$ over $Y=\mathrm{Spec}(A)\backslash\{m\}$, such that its image in $W(K)$ under the restriction map is equal to q. This completes the proof. □

Using Corollary 4.4, one can prove the following theorem (cf. [**8**, Proposition 2.1]).

Corollary 4.5. *Let A be a regular ring of dimension 3 and K be its quotient field. Assume that 2 is invertible in A. The sequence*

$$0\longrightarrow W(A)\longrightarrow W(K)\xrightarrow{\oplus\partial_P}\bigoplus_{P\in\mathrm{Spec}^1(A)} W(A_P/PA_P)$$

is exact.

We end this paper by giving a computation of $W^{-1}(Y)$ using Theorem 4.3.

Corollary 4.6. *Let A be a regular local ring of dimension 3 and m be its maximal ideal. Assume that 2 is invertible in A. Let $Y=\mathrm{Spec}(A)\backslash\{m\}$. Then $W^{-1}(Y)\simeq W(A/m)$.*

Proof. Since $W^{-1}(A)=0$ and $W_{\mathrm{fl}}(A)\simeq W(A/m)$, the result follows from Theorem 4.3. □

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