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On L^1-kernels of unitary representations of semisimple Lie groups

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1. *Introduction and statement of the results*

Let G be a locally compact group with fixed left Haar measure dx. Recall that G is said to be amenable if there exists a left translation invariant mean on the space $L^\infty(G)$, i.e. if there exists a positive, linear functional M on $L^\infty(G)$ such that $M(1_G) = 1$ and $M({}_x\phi) = M(\phi)$ for all $\phi \in L^\infty(G)$, $x \in G$, where ${}_x\phi$ denotes the left translate ${}_x\phi(y) = \phi(xy)$. The class of amenable groups includes all soluble and all compact groups (concerning the theory of amenable groups we refer to [**9**]). It is easy to see that G is amenable if and only if $\mathbb{C}1_G$, the space of the constant functions on G, has a closed left translation invariant complement in $L^\infty(G)$. This reformulation of amenability leads to the following more general question.

Let π be a unitary representation of the arbitrary group G, and let X_π be the weak $*$-closed subspace of $L^\infty(G)$ generated by the matrix coefficients of π (observe that $X_\pi = \mathbb{C}1_G$ when π is the trivial one-dimensional representation 1_G of G). When does X_π have a closed, left translation invariant complement in $L^\infty(G)$? It follows from [**1**], theorem 1, that this is the case if and only if the L^1-kernel of π, that is, the ideal $\ker_{L^1}\pi = \{f \in L^1(G) : \pi(f) = 0\}$, has bounded right approximate units. Recall that a normed algebra A has bounded right approximate units if there is a bounded net $(u_\alpha)_\alpha$ in A such that $\lim \|uu_\alpha - u\| = 0$ for all $u \in A$. In particular, G is amenable if and only if the augmentation ideal

$$L^\circ(G) = \ker_{L^1} 1_G = \{f \in L^1(G);\ \int_G f(x)\,dx = 0\}$$

has bounded right approximate units. This is Reiter's characterization of amenability (cf. [**9**], theorem 10·1).

Now let G be a non-compact, connected semisimple Lie group with finite centre. It is well known that G is not amenable. Our purpose in this paper is to prove the following stronger version of non-amenability of G: the weak $*$-closed subspace of $L^\infty(G)$ generated by the matrix coefficients of an irreducible representation π of G never has a left translation invariant, closed complement in $L^\infty(G)$ (equivalently, $\ker_{L^1}\pi$ never has bounded right approximate units). In fact we prove, in the case where G is simple, a more general result giving a description of the L^1-kernels of unitary representations of G with bounded right approximate units. Observe that such a kernel, being a $*$-ideal, has bounded right approximate units if and only if it has bounded left approximate units, and this is the case if and only if it has bounded two-sided approximate units.

Here are the precise statements of our results.

THEOREM. *Let G be a non-compact, connected simple real Lie group with finite centre Z. Let π be a strongly continuous unitary representation of G. Let X_π be the weak $*$-closed subspace of $L^\infty(G)$ generated by the matrix coefficients of π. Then the following are equivalent:*

(i) $\ker_{L^1}\pi$ *has bounded right approximate units;*

(i)′ X_π *has a left translation invariant, norm closed complement in* $L^\infty(G)$;

(ii) $\ker_{L^1}\pi = \ker_{L^1}\mathrm{ind}_Z^G\tau$ *for some unitary representation τ of Z, where* $\mathrm{ind}_Z^G\tau$ *denotes the unitary representation of G induced by τ;*

(ii)′ $X_\pi = L^\infty(G) * \mu$ *for some measure μ on G with* $\mathrm{supp}\,\mu \subseteq Z$.

In particular, if Z is trivial, then (i) *or* (i)′ *holds only if* $X_\pi = L^\infty(G)$.

COROLLARY. *Let G be a non-compact, connected semisimple real Lie group with finite centre Z. Let π be an irreducible unitary representation of G. Then, with the above notation, X_π has no left translation invariant, closed complement in $L^\infty(G)$, and* $\ker_{L^1}\pi$ *has no bounded right approximate units.*

These results are in sharp contrast to the corresponding results for some classes of amenable groups. The weak $*$-closed, translation invariant complemented subspaces of $L^\infty(G)$ for abelian G are described by means of the so-called coset ring of the dual group $\hat{G}$ (see [**4**], theorem A, [**10**], §17, theorem 2 and [**8**], theorem 13). Some partial results have been obtained in [**2**] for exponential Lie groups and for motion groups. A generalization of Reiter's characterization of amenability is given in [**13**].

It should be mentioned that the main tools for the proof of our theorem are the vanishing theorem of Howe and Moore [7] and the description of the central idempotent measures on connected locally compact groups from [**5**].

2. *Proofs*

Proof of the Theorem. As mentioned above, the equivalence of (i) and (i)′ follows from [**1**], theorem 1, since X_π is the annihilator of $\ker_{L^1}\pi$ in $L^\infty(G)$.

To show that (ii) and (ii)′ are equivalent, let τ be a unitary representation of Z. Let $I = \ker_{L^1}\mathrm{ind}_Z^G\tau$ and $J = \ker_{L^1}\tau$. Then, by [**6**], 2·2 lemma,

$$I = \{f \in L^1(G);\ {}_x f|Z \in J \text{ for almost all } x \in G\}.$$

Let $\lambda_1, \dots, \lambda_n$ be the distinct unitary characters of Z for which $J = \bigcap_{i=1}^n \ker_{L^1}\lambda_i$. Then clearly

$$I = \bigcap_{i=1}^{n} \{f \in L^1(G);\ \bar{\lambda}_i * f = 0\},$$

where functions on Z are viewed as measures on G. Now let

$$\sigma = \frac{1}{|Z|}\sum_{j=1}^{n} \bar{\lambda}_j.$$

Let $f \in L^1(G)$ with $\sigma * f = 0$. Then for $i = 1, \dots, n$ we have

$$0 = \bar{\lambda}_i * (\sigma * f) = \frac{1}{|Z|}\sum_{j=1}^{n} (\bar{\lambda}_i * \bar{\lambda}_j) * f = \bar{\lambda}_i * f,$$

by the orthogonality relations for Z. Hence

$$I = \{f \in L^1(G);\ \sigma * f = 0\}.$$

Since σ is idempotent, it follows that

$$I = (\delta_e - \sigma) * L^1(G),$$

where δ_e is the Dirac measure at the group unit e. Let $\mu = \delta_e - \bar{\sigma}$, and let $I^\perp$ be the annihilator of I in $L^\infty(G)$. Then $I^\perp = L^\infty(G) * \mu$. This shows the implication (ii) $\Rightarrow$ (ii)′. Conversely, let $\mu \neq 0$ be a function on Z and let I be the annihilator of $L^\infty(G) * \mu$ in $L^1(G)$. Then

$$I = \{f \in L^1(G);\ \check{\mu} * f = 0\},$$

where $\check{\mu}$ is the function on Z defined by $\check{\mu}(z) = \mu(z^{-1})$. Write $\check{\mu} = \sum_{i=1}^n a_i \lambda_i$ for $a_1, \ldots, a_n \in \mathbb{C}\backslash\{0\}$ and for distinct characters $\lambda_1, \ldots, \lambda_n$ of Z. Then (as above)

$$I = \bigcap_{i=1}^n \{f \in L^1(G);\ \lambda_i * f = 0\}.$$

Hence $$I = \ker_{L^1} \operatorname{ind}_Z^G \tau, \quad \text{where } \tau = \bigoplus_{i=1}^n \bar{\lambda}_i.$$

This proves the implication (ii)′ $\Rightarrow$ (ii). Notice that the above arguments also show the following: if μ is a function on Z, then $L^\infty(G) * \mu = L^\infty(G) * \tilde{\mu}$ for some idempotent function $\tilde{\mu}$ on Z. This proves (ii) $\Rightarrow$ (i)′, since then

$$L^\infty(G) \to L^\infty(G) * \tilde{\mu}, \quad f \mapsto f * \tilde{\mu}$$

is a bounded projection commuting with left translations by elements of G.

It remains to prove the implication (i)′ $\Rightarrow$ (ii)′. Let $P: L^\infty(G) \to X_\pi$ be a bounded projection onto X_π with $P({}_x\phi) = {}_xP(\phi)$ for all $\phi \in L^\infty(G), x \in G$. Let $\mathrm{UCB}(G)$ be the space of (the equivalence classes of) the right uniformly continuous bounded functions on G. Then $P(\mathrm{UCB}(G)) \subseteq \mathrm{UCB}(G)$, since $\mathrm{UCB}(G)$ consists of those $\phi \in L^\infty(G)$ for which the map

$$G \to L^\infty(G), \quad x \mapsto {}_x\phi$$

is continuous. Hence, denoting by $C_0(G)$ the space of all continuous functions on G vanishing at infinity, we can define a bounded linear functional on $C_0(G)$ by

$$C_0(G) \to \mathbb{C}, \quad \phi \mapsto (P\phi)(e).$$

Let μ be the bounded measure on G for which

$$(P\phi)(e) = \int_G \phi(y^{-1})\, d\mu(y), \quad \phi \in C_0(G).$$

Then, since P commutes with left translations, $P\phi = \phi * \mu$ for all $\phi \in C_0(G)$. Hence $X_\pi \cap C_0(G) = C_0(G) * \mu$. Moreover, μ is idempotent as P is a projection. Now $\ker_{L^1} \pi$ is invariant under the involution $f^*(x) = \overline{f(x^{-1})}$ on $L^1(G)$. Hence $\ker_{L^1} \pi$ has also bounded left approximate units. The right-hand version of [**1**], theorem 1 yields a projection $\tilde{P}: L^\infty(G) \to X_\pi$ commuting with right translations. Therefore, by the same arguments as above, $X_\pi \cap C_0(G) = \tilde{\mu} * C_0(G)$ for some idempotent measure $\tilde{\mu}$ on G. Thus

$$\tilde{\mu} * \phi = (\tilde{\mu} * \phi) * \mu = \tilde{\mu} * (\phi * \mu) = \phi * \mu$$

for all $\phi \in C_0(G)$. This implies that $\mu = \tilde{\mu}$ and μ is a central measure. By [**5**], theorem, the support of μ is contained in the largest compact normal subgroup in G, hence $\operatorname{supp}\mu \subseteq Z$. It remains to show that $X_\pi = L^\infty(G) * \mu$. Now, if π does not contain the trivial representation 1_G, then all matrix coefficients of π belong to $C_0(G)$, by the vanishing theorem of Howe and Moore (see [7], theorem 5·2). Hence, in this case, $X_\pi \cap C_0(G) = C_0(G) * \mu$ is weak $*$-dense in X_π, and this yields the conclusion.

Consider now the case where π contains 1_G. Write $\pi = \pi_0 \oplus \pi_1$, where π_0 is a multiple of 1_G and where π_1 does not contain 1_G. Then $X_\pi = X_{\pi_1} + \mathbb{C}1_G$. We claim that $\mathbb{C}1_G \subseteq X_{\pi_1}$, i.e. $X_\pi = X_{\pi_1}$, and we are reduced to the case where π does not contain 1_G. Indeed, suppose $\mathbb{C}1_G$ is not contained in X_{π_1}. Then $X_\pi = X_{\pi_1} \oplus \mathbb{C}1_G$. Let Y be a closed translation invariant complement for X_π in $L^\infty(G)$. Then $X_{\pi_1} \oplus Y$ is a closed translation invariant complement for $\mathbb{C}1_G$ in $L^\infty(G)$, a contradiction since G is not amenable. This completes the proof of the theorem. ∎

Remark. Consider the more general case of a connected, semisimple Lie group G with finite centre. The conclusion of the above theorem is probably true in this situation. The proof shows that (i), (i)′, (ii) and (ii)′ are equivalent if all the matrix coefficients of the unitary representation π of G are in $C_0(G)$. Indeed, in this case $X_\pi \cap C_0(G)$ is weak $*$-dense in X_π and this implies that $X_\pi = L^\infty(G) * \mu$, where μ is the measure defined above. This proves the crucial implication (i)′ ⇒ (ii)′. The proof of the other implications remains unchanged.

Proof of the Corollary. We first reduce to the case in which G is a direct product of simple groups. Let $\tilde{G} \to G$ be a finite covering of G. Identifying G with $\tilde{G}/N$ for some (finite) normal subgroup N, we can view $L^\infty(G)$ as a subspace of $L^\infty(\tilde{G})$. Notice that $L^\infty(G)$ has a closed translation invariant complement in $L^\infty(\tilde{G})$. Indeed, the mapping $P: L^\infty(\tilde{G}) \to L^\infty(\tilde{G})$ defined by

$$(P\phi)(x) = \frac{1}{|N|} \sum_{z \in N} \phi(xz), \quad \phi \in L^\infty(\tilde{G}),\, x \in \tilde{G}$$

is a bounded projection onto $L^\infty(G)$ commuting with translations. Suppose now the corollary is true for $\tilde{G}$. Let π be an irreducible unitary representation of G, which we view as a representation of $\tilde{G}$. Then X_π has no translation invariant complement in $L^\infty(\tilde{G})$. Hence X_π has no such complement in $L^\infty(G)$. Indeed, if Q is a projection of $L^\infty(G)$ onto $X_\pi \subseteq L^\infty(G)$ which commutes with translations, then QP is a projection of $L^\infty(\tilde{G})$ onto X_π which commutes with translations.

Consequently, we may suppose that G has the form $G = G_1 \times \ldots \times G_n$, where each G_i is a simple group with finite centre. Let π be an irreducible unitary representation of G. In view of Reiter's characterization of amenability, we may suppose $\pi \neq 1_G$. As is well known, connected semisimple Lie groups with finite centre are type I (see, e.g. [12], 4·5·7·3 and 4·5·2·11). Hence, π is a tensor product of irreducible unitary representations of the factors G_i (cf. [3], 13·1·8). Let G' be the product of the factors G_i for which $\pi \mid G_i = 1_{G_i}$, and let G'' be the product of the remaining factors. (Note that $G'' \neq \{e\}$.) Then the matrix coefficients of $\pi \mid G''$ are in $C_0(G'')$ (cf. [7], theorem 5·2).

Suppose that X_π has a closed translation invariant complement Y in $L^\infty(G)$. Then $Y \cap L^\infty(G'')$ is a closed translation invariant complement for X_π in $L^\infty(G'')$, where we identified G'' with G/G'. Hence, we may suppose that $G = G''$, that is, all matrix coefficients of π are in $C_0(G)$. Therefore (see the above remark)

$$\ker_{L^1} \pi = \ker_{L^1} \operatorname{ind}_Z^G \tau$$

for some unitary representation τ of the centre Z of G. Since π is irreducible, $\tau = \lambda I$ for the central character λ of π. Hence we may assume that $\tau = \lambda$.

Let ρ denote the induced representation $\mathrm{ind}_Z^G \lambda$. Recall that the Hilbert space H_ρ of ρ is the space of all square-integrable measurable functions $\xi: G \to \mathbb{C}$ which satisfy the equality

$$\xi(xz) = \bar{\lambda}(z)\,\xi(x), \quad \text{for } z \in Z \text{ and almost all } x \in G.$$

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{A}$ be the universal enveloping algebra of the complexification of $\mathfrak{g}$. Let $\mathfrak{A}$ be realized as the algebra of the left invariant differential operators on G. Let π_∞ and ρ_∞ denote the representations of $\mathfrak{A}$ corresponding to π and ρ on the spaces H_π^∞ and H_ρ^∞ of the C^∞-vectors of H_π and H_ρ, respectively. We want to show that $\ker \rho_\infty = 0$ while $\ker \pi_\infty \neq 0$. Indeed, fix some neighbourhood U of e such that $zU \cap U = \emptyset$ for each $z \in Z, z \neq e$. Then each $\phi \in C^\infty(G)$ with $\operatorname{supp}\phi \subseteq U$ coincides on U with some $\tilde{\phi} \in H_\rho^\infty$: take simply $\tilde{\phi}$ defined by

$$\tilde{\phi}(x) = \sum_{z \in Z} \phi(xz)\,\lambda(z), \quad x \in G.$$

It follows from this that, if $D \in \ker \rho_\infty$, then $D\phi = 0$ for all $\phi \in C^\infty(G)$ and hence $D = 0$.

On the other hand, let $\mathfrak{J}$ be the centre of $\mathfrak{A}$, and let $\mathfrak{X}_\pi$ be the infinitesimal character of π, that is, $\mathfrak{X}_\pi$ is the character of $\mathfrak{J}$ for which $\pi_\infty(D) = \mathfrak{X}_\pi(D)\,I$ for each $D \in \mathfrak{J}$. It is known that $\mathfrak{J}$ has infinite dimension (in fact, $\mathfrak{J}$ is an algebra of polynomials, see [**11**], 4·9·3 and 4·10·3). Hence $\mathfrak{X}_\pi(D) = 0$ for some $D \in \mathfrak{J}$, $D \neq 0$. Therefore $\ker \pi_\infty \neq 0$.

Now, $\sigma(D\phi) = \sigma_\infty(D)\,\sigma(\phi)$ for each unitary representation σ of G, for each $D \in \mathfrak{A}$ and for each $\phi \in \mathfrak{D}(G)$, the space of all infinitely differentiable functions on G with compact support. Take some $D \in \ker \pi_\infty$, $D \neq 0$. Then $D\phi \in \ker_{L^1} \pi$ and hence $D\phi \in \ker_{L^1} \rho$ for each $\phi \in \mathfrak{D}(G)$. Let $(\phi_\alpha)_\alpha$ be an approximate unit for $L^1(G)$ with functions ϕ_α from $\mathfrak{D}(G)$. Then

$$0 = \rho(D\phi_\alpha) = \rho_\infty(D)\,\rho(\phi_\alpha)$$

for all α. Since $\rho(\phi_\alpha)$ converges strongly to I, this implies $\rho_\infty(D) = 0$, a contradiction to $\ker \rho_\infty = 0$. This shows that X_π has no closed, translation invariant complement. ∎

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