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Density of Rational Points on a Certain Smooth Bihomogeneous Threefold

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We establish sharp upper and lower bounds for the number of rational points of bounded anticanonical height on a smooth bihomogeneous threefold defined over $\mathbb{Q}$ and of bidegree $(1, 2)$. These bounds are in agreement with Manin's conjecture.

1 Introduction

Let $n \geq 2$ and $d \geq 1$ be two integers such that $n \geq d$. Let $V_d^n \subset \mathbb{P}^n \times \mathbb{P}^n$ be the smooth hypersurface defined over a number field K by the equation

$$x_0 y_0^d + \cdots + x_n y_n^d = 0,$$

where we use the notation $(\mathbf{x}, \mathbf{y}) = ((x_0 : \cdots : x_n), (y_0 : \cdots : y_n))$ to denote the coordinates in the biprojective space $\mathbb{P}^n \times \mathbb{P}^n$.

The family of smooth bihomogeneous varieties V_d^n is an excellent testing ground for the validity of Manin's conjecture on the asymptotic behavior of the number of rational points of bounded anticanonical height on Fano varieties (see [4]). For instance, Batyrev and Tschinkel have provided a famous counterexample to this conjecture in the case $n = 3$, $d = 3$, and under the assumption that K contains a nontrivial cube root of unity.

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From now on, we focus on the case $K=\mathbb{Q}$. We define the usual exponential height function $H:\mathbb{P}^n(\mathbb{Q})\to\mathbb{R}_{>0}$ as follows. Given $\mathbf{z}\in\mathbb{P}^n(\mathbb{Q})$, we can choose coordinates $(z_0:\dots:z_n)$ satisfying $(z_0,\dots,z_n)\in\mathbb{Z}^{n+1}$ and $\gcd(z_0,\dots,z_n)=1$, and then we can set

$$H(\mathbf{z})=\max\{|z_i|, i=0,\dots,n\}.$$

With this in mind, we can define a height function $\mathbf{H}:\mathbb{P}^n(\mathbb{Q})\times\mathbb{P}^n(\mathbb{Q})\to\mathbb{R}_{>0}$ by setting

$$\mathbf{H}(\mathbf{x},\mathbf{y})=H(\mathbf{x})^nH(\mathbf{y})^{n+1-d},$$

for $(\mathbf{x},\mathbf{y})\in\mathbb{P}^n(\mathbb{Q})\times\mathbb{P}^n(\mathbb{Q})$. For any Zariski open subset U_d^n of V_d^n, we can introduce the number of rational points of bounded anticanonical height on U_d^n, that is

$$N_{U_d^n,\mathbf{H}}(B)=\#\{(\mathbf{x},\mathbf{y})\in U_d^n(\mathbb{Q}), \mathbf{H}(\mathbf{x},\mathbf{y})\leq B\}.$$

In this setting, Manin's conjecture predicts that there should exist an open subset U_d^n of V_d^n such that

$$N_{U_d^n,\mathbf{H}}(B)=cB\log B(1+o(1)), \tag{1.1}$$

where $c>0$ is a constant depending on V_d^n and $\mathbf{H}$, and which is expected to obey Peyre's prediction [7]. As already mentioned, this conjecture is known not to hold in such generality.

Let us mention that more generally, the term $\log B$ in conjecture (1.1) is expected to appear to the power $\rho-1$ where ρ denotes the rank of the Picard group of the variety considered. A proof of the fact that the Picard group of V_d^n is $\mathbb{Z}^2$ can be found in [10, Theorem 2.4].

The circle method is a traditional technique to count solutions to diophantine equations, and it has recently been applied by Schindler [9, 10] to count rational points on bihomogeneous varieties. In particular, [10, Theorem 1.2] states that smooth hypersurfaces in biprojective space $\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}$ defined by general bihomogeneous forms of bidegree (d_1,d_2) satisfy Manin's conjecture provided that $d_1,d_2\geq 2$ and $\min\{n_1,n_2\}>3\cdot 2^{d_1+d_2}d_1d_2$.

Similarly, the circle method is only expected to yield a proof of Manin's conjecture for V_d^n if n is exponentially large in terms of d.

It is natural to start by investigating the cases where d is small. If $d=1$, then V_1^n satisfies Manin's conjecture for any $n\geq 2$. This follows from the result of Franke et al. [4] on flag varieties, which makes use of the work of Langlands about the meromorphic continuation of Eisenstein series. Other proofs have then been obtained by many authors using a great variety of techniques (see [2, 8, 11, 12]).

The next case of interest is $d=2$. Here, there is nothing written down in the literature and, in particular, the cases $n=2$ and $n=3$ are known to be extremely hard problems. The aim of this article is to investigate what can be achieved in the case $n=2$.

Unfortunately, we are unable to establish Manin's conjecture for V_2^2. However, we are able to prove upper and lower bounds of the exact order of magnitude for $N_{U_2^2,\mathbf{H}}(B)$, where U_2^2 is the open subset defined by removing from V_2^2 the subset given by $x_0x_1x_2y_0y_1y_2=0$.

Our main result is the following theorem.

Theorem 1. We have the bounds

$$B\log B \ll N_{U_2^2,\mathbf{H}}(B) \ll B\log B. \qquad \square$$

It is worth emphasizing that these bounds are in agreement with the prediction of Manin (1.1).

Let us give a sketch of the proof. In what follows, we denote by $\varphi_i:\mathbb{P}^2\times\mathbb{P}^2\to\mathbb{P}^2$, $i\in\{1,2\}$, the two projections.

First, we remark that proving the lower bound is not hard. Indeed, it suffices to note that the contribution to $N_{U_2^2,\mathbf{H}}(B)$ of the fibers of φ_2 corresponding to rational points $\mathbf{y}\in\mathbb{P}^2(\mathbb{Q})$ whose height is bounded by a small power of B is of the expected order of magnitude. This is achieved in Section 3.

The proof of the upper bound is more intricate. It mainly relies on Lemma 4 which gives an upper bound for the number of solutions to a slightly more general equation than $x_0y_0^2+x_1y_1^2+x_2y_2^2=0$. To prove this lemma, we make use of both geometry of numbers and analytic number theory results.

More specifically, we get a first upper bound by estimating the number of $\mathbf{x}\in\mathbb{P}^2(\mathbb{Q})$ for fixed $\mathbf{y}\in\mathbb{P}^2(\mathbb{Q})$ and by summing trivially over the fibers of φ_2. Similarly, we obtain a second upper bound by estimating the number of $\mathbf{y}\in\mathbb{P}^2(\mathbb{Q})$ for fixed $\mathbf{x}\in\mathbb{P}^2(\mathbb{Q})$. However, it is worth noting that the summation over the fibers of φ_1 has to be carried out nontrivially because we need to take advantage of the fact that most diagonal conics do not have a rational point. To complete the proof, it only remains to minimize these two upper bounds, basically depending on the respective sizes of $\mathbf{x}$ and $\mathbf{y}$.

Finally, it is worth mentioning that Lemma 4 will be useful in other settings. In particular, it plays a crucial role in the work of the author [6], where it is proved that the number of rational points of bounded height on certain elliptic fibrations grows linearly, as predicted by Manin's conjecture.

2 Geometry of Numbers

We now recall two lemmas that provide upper bounds for the number of solutions to certain homogeneous diagonal equations in three variables and constrained in boxes. The first of these two lemmas is concerned with the case of a linear equation and is due to Heath-Brown [5, Lemma 3].

Lemma 1. Let $\mathbf{w} = (w_0, w_1, w_2) \in \mathbb{Z}^3$ be a primitive vector and let $U_i \geq 1$ for $i \in \{0, 1, 2\}$. Let also $N_{\mathbf{w}} = N_{\mathbf{w}}(U_0, U_1, U_2)$ be the number of primitive vectors $(u_0, u_1, u_2) \in \mathbb{Z}^3$ satisfying $|u_i| \leq U_i$ for $i \in \{0, 1, 2\}$ and the equation

$$u_0 w_0 + u_1 w_1 + u_2 w_2 = 0.$$

We have the bound

$$N_{\mathbf{w}} \leq 12\pi \frac{U_0 U_1 U_2}{\max\{|w_i| U_i\}} + 4,$$

where the maximum is taken over $i \in \{0, 1, 2\}$. In particular, if $\mathbf{w} \in \mathbb{Z}^3_{\neq 0}$, then

$$N_{\mathbf{w}} \ll \frac{(U_0 U_1 U_2)^{2/3}}{|w_0 w_1 w_2|^{1/3}} + 1. \qquad \square$$

The second lemma deals with the case of a quadratic equation and immediately follows from the result of Browning and Heath-Brown [3, Corollary 2].

Lemma 2. Let $\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{Z}^3_{\neq 0}$ be a vector satisfying the conditions $\gcd(u_i, u_j) = 1$ for $i, j \in \{0, 1, 2\}$, $i \neq j$, and let $V_i \geq 1$ for $i \in \{0, 1, 2\}$. Let also $N_{\mathbf{u}} = N_{\mathbf{u}}(V_0, V_1, V_2)$ be the number of primitive vectors $(v_0, v_1, v_2) \in \mathbb{Z}^3$ satisfying $|v_i| \leq V_i$ for $i \in \{0, 1, 2\}$ and the equation

$$u_0 v_0^2 + u_1 v_1^2 + u_2 v_2^2 = 0.$$

We have the bound

$$N_{\mathbf{u}} \ll \left(\frac{V_0 V_1 V_2}{|u_0 u_1 u_2|} + 1 \right)^{1/3} \tau(|u_0 u_1 u_2|). \qquad \square$$

We also need to consider how often a diagonal quadratic equation has a nontrivial integral solution. For this, we recall the following lemma, which is a particular case of the nice result of Browning [1, Proposition 1]. Let us note that this result is deep and builds upon several powerful analytic number theory tools.

Lemma 3. Let $\mathbf{f} = (f_0, f_1, f_2) \in \mathbb{Z}^3_{\neq 0}$ be a primitive vector and let $U_i \geq 1$ for $i \in \{0, 1, 2\}$. Let also $\mathcal{T}_{\mathbf{f}}(U_0, U_1, U_2)$ be the set of $\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{Z}^3_{\neq 0}$ satisfying $|u_i| \leq U_i$ for $i \in \{0, 1, 2\}$, and

$\gcd(u_i, u_j) = 1$ for $i, j \in \{0, 1, 2\}$, $i \neq j$, and such that the equation

$$f_0 u_0 v_0^2 + f_1 u_1 v_1^2 + f_2 u_2 v_2^2 = 0,$$

has a solution $(v_0, v_1, v_2) \in \mathbb{Z}_{\neq 0}^3$ with $\gcd(v_i, v_j) = 1$ for $i, j \in \{0, 1, 2\}$, $i \neq j$. Let $\varepsilon > 0$ be fixed. We have the bound

$$\sum_{\mathbf{u} \in \mathcal{T}_{\mathbf{f}}(U_0, U_1, U_2)} 2^{\omega(|u_0 u_1 u_2|)} \ll |f_0 f_1 f_2|^{\varepsilon} U_0 U_1 U_2 M_{\varepsilon}(U_0, U_1, U_2),$$

where

$$M_{\varepsilon}(U_0, U_1, U_2) = 1 + \max_{\{i,j,k\} = \{0,1,2\}} (U_i U_j)^{-1/2+\varepsilon} \log 2U_k. \qquad \square$$

These three lemmas together allow us to prove a sharp upper bound for the number of solutions $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}_{\neq 0}^3 \times \mathbb{Z}_{\neq 0}^3$ to the equation of Lemma 3 and constrained in boxes. More precisely, we establish the following lemma, which is the key result in the proof of the upper bound in Theorem 1.

Lemma 4. Let $\mathbf{f} = (f_0, f_1, f_2) \in \mathbb{Z}_{\neq 0}^3$ be a vector satisfying the conditions $\gcd(f_i, f_j) = 1$ for $i, j \in \{0, 1, 2\}$, $i \neq j$, and let $U_i, V_i \geq 1$ for $i \in \{0, 1, 2\}$. Let also $N_{\mathbf{f}} = N_{\mathbf{f}}(U_0, U_1, U_2, V_0, V_1, V_2)$ be the number of vectors $(u_0, u_1, u_2) \in \mathbb{Z}_{\neq 0}^3$ and $(v_0, v_1, v_2) \in \mathbb{Z}_{\neq 0}^3$ satisfying $|u_i| \leq U_i$, $|v_i| \leq V_i$ for $i \in \{0, 1, 2\}$, and the equation

$$f_0 u_0 v_0^2 + f_1 u_1 v_1^2 + f_2 u_2 v_2^2 = 0,$$

and such that $\gcd(u_i v_i, u_j v_j) = 1$ for $i, j \in \{0, 1, 2\}$, $i \neq j$. Let $\varepsilon > 0$ be fixed and recall the definition of $M_{\varepsilon}(U_0, U_1, U_2)$ given in Lemma 3. We have the bound

$$N_{\mathbf{f}} \ll |f_0 f_1 f_2|^{\varepsilon} (U_0 U_1 U_2)^{2/3} (V_0 V_1 V_2)^{1/3} M_{\varepsilon}(U_0, U_1, U_2). \qquad \square$$

Proof. First, let us fix $(v_0, v_1, v_2) \in \mathbb{Z}_{\neq 0}^3$ and let us start by bounding the number of $(u_0, u_1, u_2) \in \mathbb{Z}_{\neq 0}^3$ satisfying the conditions stated in the lemma. Since $\gcd(f_0 v_0^2, f_1 v_1^2, f_2 v_2^2) = 1$, Lemma 1 gives

$$N_{\mathbf{f}} \ll \sum_{\substack{|v_i| \leq V_i \\ i \in \{0,1,2\}}} \left(\frac{1}{|f_0 f_1 f_2|^{1/3}} \frac{(U_0 U_1 U_2)^{2/3}}{|v_0 v_1 v_2|^{2/3}} + 1 \right).$$

In particular, this gives us a first upper bound

$$N_{\mathbf{f}} \ll (U_0 U_1 U_2)^{2/3} (V_0 V_1 V_2)^{1/3} + V_0 V_1 V_2. \tag{2.1}$$

In a similar fashion, let us fix $(u_0, u_1, u_2) \in \mathbb{Z}_{\neq 0}^3$ and let us start by bounding the number of $(v_0, v_1, v_2) \in \mathbb{Z}_{\neq 0}^3$ satisfying the conditions stated in the lemma. The equation

$$f_0 u_0 v_0^2 + f_1 u_1 v_1^2 + f_2 u_2 v_2^2 = 0,$$

and the coprimality conditions $\gcd(f_i, f_j) = \gcd(u_i v_i, u_j v_j) = 1$ for $i, j \in \{0, 1, 2\}$, $i \neq j$, imply that $\gcd(f_i u_i, f_j u_j) = 1$ for $i, j \in \{0, 1, 2\}$, $i \neq j$. We can thus apply Lemma 2. Recalling the notation introduced in Lemma 3, we obtain

$$N_{\mathbf{f}} \ll \sum_{\mathbf{u} \in \mathcal{T}_{\mathbf{f}}(U_0, U_1, U_2)} \left(\frac{1}{|f_0 f_1 f_2|^{1/3}} \frac{(V_0 V_1 V_2)^{1/3}}{|u_0 u_1 u_2|^{1/3}} + 1 \right) \tau(|f_0 f_1 f_2 u_0 u_1 u_2|).$$

This implies in particular that

$$N_{\mathbf{f}} \ll |f_0 f_1 f_2|^{\varepsilon} \sum_{\mathbf{u} \in \mathcal{T}_{\mathbf{f}}(U_0, U_1, U_2)} \left(\frac{(V_0 V_1 V_2)^{1/3}}{|u_0 u_1 u_2|^{1/3}} + 1 \right) \tau(|u_0 u_1 u_2|).$$

Let us write $u_i = z_i^2 \ell_i$ with $z_i \in \mathbb{Z}_{>0}$ and $|\mu(|\ell_i|)| = 1$ for $i \in \{0, 1, 2\}$, and let us set $\mathbf{l} = (\ell_0, \ell_1, \ell_2)$, $\mathbf{g} = (f_0 z_0^2, f_1 z_1^2, f_2 z_2^2)$ and $L_i = U_i / z_i^2$ for $i \in \{0, 1, 2\}$. We have

$$N_{\mathbf{f}} \ll |f_0 f_1 f_2|^{\varepsilon} \sum_{\substack{z_i \leq U_i^{1/2} \\ i \in \{0,1,2\}}} (z_0 z_1 z_2)^{\varepsilon} \sum_{\substack{\mathbf{l} \in \mathcal{T}_{\mathbf{g}}(L_0, L_1, L_2) \\ |\mu(|\ell_0 \ell_1 \ell_2|)| = 1}} \left(\frac{(V_0 V_1 V_2)^{1/3}}{(z_0 z_1 z_2)^{2/3} |\ell_0 \ell_1 \ell_2|^{1/3}} + 1 \right) 2^{\omega(|\ell_0 \ell_1 \ell_2|)}.$$

Note that we have used the fact that $\ell_0 \ell_1 \ell_2$ is squarefree to replace the arithmetic function τ by 2^{ω}. Let $\varepsilon > 0$ be fixed. We note that $\mathbf{g}$ is primitive so we can use Lemma 3. Thus, applying partial summation and Lemma 3, we get

$$N_{\mathbf{f}} \ll |f_0 f_1 f_2|^{2\varepsilon} \sum_{\substack{z_i \leq U_i^{1/2} \\ i \in \{0,1,2\}}} \frac{(U_0 U_1 U_2)^{2/3} (V_0 V_1 V_2)^{1/3} + U_0 U_1 U_2}{(z_0 z_1 z_2)^{2 - 3\varepsilon}} M_{\varepsilon}(L_0, L_1, L_2).$$

Using the trivial inequality

$$M_{\varepsilon}(L_0, L_1, L_2) \leq 1 + \sum_{\{i,j,k\} = \{0,1,2\}} (U_i U_j)^{-1/2 + \varepsilon} (z_i z_j)^{1 - 2\varepsilon} \log 2U_k,$$

we obtain

$$\sum_{\substack{z_i \leq U_i^{1/2} \\ i \in \{0,1,2\}}} \frac{M_{\varepsilon}(L_0, L_1, L_2)}{(z_0 z_1 z_2)^{2 - 3\varepsilon}} \ll M_{2\varepsilon}(U_0, U_1, U_2).$$

This finally gives us a second upper bound

$$N_{\mathbf{f}} \ll |f_0 f_1 f_2|^{2\varepsilon} ((U_0 U_1 U_2)^{2/3} (V_0 V_1 V_2)^{1/3} + U_0 U_1 U_2) M_{2\varepsilon}(U_0, U_1, U_2). \tag{2.2}$$

As a result, putting together the upper bounds (2.1) and (2.2), and rescaling ε, we find in particular that

$$N_{\mathbf{f}} \ll |f_0 f_1 f_2|^{\varepsilon}((U_0U_1U_2)^{2/3}(V_0V_1V_2)^{1/3} + \min\{U_0U_1U_2, V_0V_1V_2\})M_{\varepsilon}(U_0, U_1, U_2).$$

The simple observation that

$$\min\{U_0U_1U_2, V_0V_1V_2\} \leq (U_0U_1U_2)^{2/3}(V_0V_1V_2)^{1/3},$$

completes the proof. ■

3 The Lower Bound

This section is devoted to the proof of the lower bound in Theorem 1. As stated in Section 1, the proof merely draws upon the fact that the contribution to $N_{U_2^2,\mathbf{H}}(B)$ of the $\mathbf{y} \in \mathbb{P}^2(\mathbb{Q})$ whose height is bounded by a small power of B is already of the expected order of magnitude.

By definition of $N_{U_2^2,\mathbf{H}}(B)$, we have

$$N_{U_2^2,\mathbf{H}}(B) = 2\#\left\{(\mathbf{x},\mathbf{y}) \in \mathbb{Z}^3_{\neq 0} \times \mathbb{Z}^3_{>0}, \begin{array}{c} x_0y_0^2 + x_1y_1^2 + x_2y_2^2 = 0 \\ \gcd(x_0,x_1,x_2) = \gcd(y_0,y_1,y_2) = 1 \\ \max_{i,j \in \{0,1,2\}} x_i^2 y_j \leq B \end{array}\right\}.$$

It is convenient to note that we thus have

$$N_{U_2^2,\mathbf{H}}(B) \geq 12 \sum_{\substack{\mathbf{y} \in \mathbb{Z}^3_{>0} \\ \gcd(y_0,y_2)=1 \\ y_0<y_1<y_2 \leq B^{1/6}}} \#\left\{\mathbf{x} \in \mathbb{Z}^3_{\neq 0}, \begin{array}{c} x_0y_0^2 + x_1y_1^2 + x_2y_2^2 = 0 \\ \gcd(x_0,x_1,x_2) = 1 \\ \max_{i \in \{0,1,2\}} x_i^2 y_2 \leq B \end{array}\right\}.$$

Since the conditions $\max_{i\in\{0,1\}} x_i^2 y_2 \leq B/4$ and $y_0, y_1 < y_2$ together with the equation $x_0y_0^2 + x_1y_1^2 + x_2y_2^2 = 0$ imply that $\max_{i\in\{0,1,2\}} x_i^2 y_2 \leq B$, we have

$$N_{U_2^2,\mathbf{H}}(B) \geq 12 \sum_{\substack{\mathbf{y} \in \mathbb{Z}^3_{>0} \\ \gcd(y_0,y_2)=1 \\ y_0<y_1<y_2 \leq B^{1/6}}} \#\left\{\mathbf{x} \in \mathbb{Z}^3_{\neq 0}, \begin{array}{c} x_0y_0^2 + x_1y_1^2 + x_2y_2^2 = 0 \\ \gcd(x_0,x_1,x_2) = 1 \\ \max_{i \in \{0,1\}} x_i^2 y_2 \leq B/4 \end{array}\right\}.$$

We can now remove the coprimality condition $\gcd(x_0, x_1, x_2) = 1$ using a Möbius inversion. We get

$$N_{U_2^2,\mathbf{H}}(B) \geq 12 \sum_{\substack{\mathbf{y}\in\mathbb{Z}_{>0}^3 \\ \gcd(y_0,y_2)=1 \\ y_0<y_1<y_2\leq B^{1/6}}} \sum_{k\leq B^{1/2}} \mu(k)\mathcal{S}_k(\mathbf{y}; B), \tag{3.1}$$

where

$$\mathcal{S}_k(\mathbf{y}; B) = \# \left\{ \mathbf{x}' \in \mathbb{Z}_{\neq 0}^3, \begin{array}{l} x_0' y_0^2 + x_1' y_1^2 + x_2' y_2^2 = 0 \\ \max_{i\in\{0,1\}} x_i'^2 y_2 \leq B/4k^2 \end{array} \right\},$$

and where we have used the obvious notation $\mathbf{x}' = (x_0', x_1', x_2')$. We now observe that

$$\mathcal{S}_k(\mathbf{y}; B) = \# \left\{ (x_0', x_1') \in \mathbb{Z}_{\neq 0}^2, \begin{array}{l} x_0' y_0^2 + x_1' y_1^2 = 0 \pmod{y_2^2} \\ \max_{i\in\{0,1\}} x_i'^2 y_2 \leq B/4k^2 \end{array} \right\} + O\left(\frac{B^{1/2}}{k y_2^{1/2}} \right).$$

Since $\gcd(y_0, y_2) = 1$, y_0 is invertible modulo y_2^2. Using the notation y_0^{-1} to denote the inverse of y_0 modulo y_2^2, we have

$$\begin{aligned} \mathcal{S}_k(\mathbf{y}; B) &= \sum_{\substack{x_1'\in\mathbb{Z}_{\neq 0} \\ x_1'^2 y_2\leq B/4k^2}} \# \left\{ x_0' \in \mathbb{Z}_{\neq 0}, \begin{array}{l} x_0' = -y_0^{-2} x_1' y_1^2 \pmod{y_2^2} \\ x_0'^2 y_2 \leq B/4k^2 \end{array} \right\} + O\left(\frac{B^{1/2}}{k y_2^{1/2}} \right) \\ &= \sum_{\substack{x_1'\in\mathbb{Z}_{\neq 0} \\ x_1'^2 y_2\leq B/4k^2}} \left(\frac{B^{1/2}}{k y_2^{5/2}} + O(1) \right) + O\left(\frac{B^{1/2}}{k y_2^{1/2}} \right) \\ &= \frac{B}{k^2 y_2^3} + O\left(\frac{B^{1/2}}{k y_2^{1/2}} \right). \end{aligned}$$

Recalling the lower bound (3.1), we see that we have obtained

$$N_{U_2^2,\mathbf{H}}(B) \geq 12 \sum_{\substack{\mathbf{y}\in\mathbb{Z}_{>0}^3 \\ \gcd(y_0,y_2)=1 \\ y_0<y_1<y_2\leq B^{1/6}}} \sum_{k\leq B^{1/2}} \mu(k) \left(\frac{B}{k^2 y_2^3} + O\left(\frac{B^{1/2}}{k y_2^{1/2}} \right) \right).$$

This eventually gives

$$N_{U_2^2,\mathbf{H}}(B) \gg B \log B,$$

which completes the proof of the lower bound in Theorem 1.

4 The Upper Bound

This section is concerned with establishing the upper bound in Theorem 1. As already explained in Section 1, the proof draws upon Lemma 4.

4.1 Parametrization of the variables

The following lemma provides us with a convenient parametrization of the rational points on U_2^2.

Lemma 5. Let $\mathcal{T}(B)$ be the number of $(f_0, f_1, f_2, g_0, g_1, g_2, h_0, h_1, h_2) \in \mathbb{Z}_{>0}^9$ and $(u_0, u_1, u_2, v_0, v_1, v_2) \in \mathbb{Z}_{\neq 0}^6$ satisfying the equation

$$f_0 u_0 v_0^2 + f_1 u_1 v_1^2 + f_2 u_2 v_2^2 = 0,$$

and the conditions $\gcd(f_i, f_j g_j h_i u_i v_j) = \gcd(g_i, g_j h_i u_i v_i v_j) = \gcd(h_i, h_j v_i) = 1$ and $\gcd(u_i, u_j) = \gcd(v_i, v_j) = 1$ for $i, j \in \{0, 1, 2\}$, $i \neq j$, and the height conditions

$$\left(\max_{\{i,j,k\}=\{0,1,2\}} f_j f_k g_j^2 g_k^2 h_i^2 |u_i| \right)^2 \left(\max_{\{i,j,k\}=\{0,1,2\}} f_i g_i h_j h_k |v_i| \right) \leq B.$$

We have the equality

$$N_{U_2^2,\mathbf{H}}(B) = \tfrac{1}{4}\mathcal{T}(B).$$

□

Proof. We have

$$N_{U_2^2,\mathbf{H}}(B) = \frac{1}{4}\# \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_{\neq 0}^3 \times \mathbb{Z}_{\neq 0}^3, \begin{array}{c} x_0 y_0^2 + x_1 y_1^2 + x_2 y_2^2 = 0 \\ \gcd(x_0, x_1, x_2) = \gcd(y_0, y_1, y_2) = 1 \\ \max_{i,j \in \{0,1,2\}} x_i^2 |y_j| \leq B \end{array} \right\}.$$

For $\{i, j, k\} = \{0, 1, 2\}$, let us set $h_i = \gcd(y_j, y_k)$ and let us write $y_i = h_j h_k y_i'$. The equation

$$x_0 h_1^2 h_2^2 y_0'^2 + x_1 h_0^2 h_2^2 y_1'^2 + x_2 h_0^2 h_1^2 y_2'^2 = 0$$

implies that for $i \in \{0, 1, 2\}$, we have $h_i^2 \mid x_i$ so that we can write $x_i = h_i^2 x_i'$. We thus obtain the equation

$$x_0' y_0'^2 + x_1' y_1'^2 + x_2' y_2'^2 = 0.$$

For $\{i, j, k\} = \{0, 1, 2\}$, let us set $X_i = \gcd(x_j', x_k')$ and let us write $x_i' = X_j X_k u_i$. We obtain

$$X_1 X_2 u_0 y_0'^2 + X_0 X_2 u_1 y_1'^2 + X_0 X_1 u_2 y_2'^2 = 0,$$

so that, for $i \in \{0, 1, 2\}$, we have $X_i \mid y_i'^2$. As a consequence, for $i \in \{0, 1, 2\}$, there is a unique way to write $X_i = f_i g_i^2$ and $y_i' = f_i g_i v_i$ for $f_i, g_i \in \mathbb{Z}_{>0}$ with $\gcd(g_i, v_i) = 1$. Therefore, we obtain the equation

$$f_0 u_0 v_0^2 + f_1 u_1 v_1^2 + f_2 u_2 v_2^2 = 0,$$

and it is not hard to check that the variables satisfy the coprimality conditions listed in the statement of the lemma, which completes the proof. ■

4.2 Proof of the upper bound

First, we note that the coprimality conditions $\gcd(f_i, v_j) = \gcd(u_i, u_j) = \gcd(v_i, v_j) = 1$ for $i, j \in \{0, 1, 2\}$, $i \neq j$, and the equation

$$f_0 u_0 v_0^2 + f_1 u_1 v_1^2 + f_2 u_2 v_2^2 = 0,$$

imply that we actually have $\gcd(u_i v_i, u_j v_j) = 1$ for $i, j \in \{0, 1, 2\}$, $i \neq j$.

For $i \in \{0, 1, 2\}$, let $F_i, G_i, H_i, U_i, V_i \geq 1/2$ run over powers of 2 and let $\mathcal{M}$ be the number of $(f_0, f_1, f_2, g_0, g_1, g_2, h_0, h_1, h_2) \in \mathbb{Z}_{>0}^9$ and $(u_0, u_1, u_2, v_0, v_1, v_2) \in \mathbb{Z}_{\neq 0}^6$ satisfying the equation

$$f_0 u_0 v_0^2 + f_1 u_1 v_1^2 + f_2 u_2 v_2^2 = 0,$$

the conditions $F_i < f_i \leq 2F_i$, $G_i < g_i \leq 2G_i$, $H_i < h_i \leq 2H_i$, $U_i < |u_i| \leq 2U_i$ and $V_i < |v_i| \leq 2V_i$, and $\gcd(f_i, f_j) = \gcd(u_i v_i, u_j v_j) = 1$ for $i, j \in \{0, 1, 2\}$, $i \neq j$. By Lemma 5, we have

$$N_{U_2^2, \mathbf{H}}(B) \ll \sum_{\substack{F_i, G_i, H_i, U_i, V_i \\ i \in \{0,1,2\}}} \mathcal{M},$$

where the sum is taken over the F_i, G_i, H_i, U_i, V_i, $i \in \{0, 1, 2\}$, satisfying

$$\left(\max_{\{i,j,k\}=\{0,1,2\}} F_j F_k G_j^2 G_k^2 H_i^2 U_i \right)^2 \left(\max_{\{i,j,k\}=\{0,1,2\}} F_i G_i H_j H_k V_i \right) \leq B. \tag{4.1}$$

By choosing $\varepsilon = 1/6$ in Lemma 4, we get

$$\mathcal{M} \ll (F_0 F_1 F_2)^{7/6} G_0 G_1 G_2 H_0 H_1 H_2 (U_0 U_1 U_2)^{2/3} (V_0 V_1 V_2)^{1/3} M_{1/6}(U_0, U_1, U_2).$$

Recalling the definition of $M_{1/6}(U_0, U_1, U_2)$ given in Lemma 3, we define

$$\mathcal{M}_1 = (F_0 F_1 F_2)^{7/6} G_0 G_1 G_2 H_0 H_1 H_2 (U_0 U_1 U_2)^{2/3} (V_0 V_1 V_2)^{1/3},$$

and

$$\mathcal{M}_2 = (\log B)(F_0F_1F_2)^{7/6}G_0G_1G_2H_0H_1H_2(U_0U_1U_2)^{2/3}(V_0V_1V_2)^{1/3} \times \left(\min_{i,j\in\{0,1,2\},i\neq j} U_iU_j\right)^{-1/3},$$

and also

$$\mathcal{N}_\ell(B) = \sum_{\substack{F_i,G_i,H_i,U_i,V_i\\ i\in\{0,1,2\}}} \mathcal{M}_\ell,$$

for $\ell \in \{1, 2\}$, and where the sum is taken over the F_i, G_i, H_i, U_i, V_i, $i \in \{0, 1, 2\}$, satisfying the conditions (4.1). We thus have

$$N_{U_2^2,\mathbf{H}}(B) \ll \mathcal{N}_1(B) + \mathcal{N}_2(B). \tag{4.2}$$

Let us start by taking care of $\mathcal{N}_1(B)$. For this, let us sum over V_0, V_1, and V_2 using the conditions (4.1). We obtain

$$\mathcal{N}_1(B) \ll B \sum_{\substack{F_i,G_i,H_i,U_i\\ i\in\{0,1,2\}}} (F_0F_1F_2)^{5/6}(G_0G_1G_2)^{2/3}(H_0H_1H_2)^{1/3}(U_0U_1U_2)^{2/3} \times \left(\max_{\{i,j,k\}=\{0,1,2\}} F_jF_kG_j^2G_k^2H_i^2U_i\right)^{-2}.$$

By symmetry, we can assume that

$$\max\{F_0F_2G_0^2G_2^2H_1^2U_1,\, F_0F_1G_0^2G_1^2H_2^2U_2\} \leq F_1F_2G_1^2G_2^2H_0^2U_0. \tag{4.3}$$

Let us sum over U_1 and U_2 using the inequalities (4.3). We obtain

$$\mathcal{N}_1(B) \ll B \sum_{\substack{F_i,G_i,H_i,U_0\\ i\in\{0,1,2\}}} (F_0F_1F_2)^{-1/2}(G_0G_1G_2)^{-2}(H_0H_1H_2)^{-1},$$

which finally gives

$$\mathcal{N}_1(B) \ll B \log B. \tag{4.4}$$

Let us now deal with $\mathcal{N}_2(B)$. We can assume by symmetry that

$$\min_{i,j\in\{0,1,2\},i\neq j} U_iU_j = U_1U_2.$$

We thus have

$$\mathcal{M}_2 \ll (\log B)(F_0F_1F_2)^{7/6}G_0G_1G_2H_0H_1H_2U_0^{2/3}(U_1U_2)^{1/3}(V_0V_1V_2)^{1/3}.$$

Once again, let us sum over V_0, V_1, and V_2 using the conditions (4.1). We find that

$$\mathcal{N}_2(B) \ll B(\log B) \sum_{\substack{F_i, G_i, H_i, U_i \\ i \in \{0,1,2\}}} (F_0 F_1 F_2)^{5/6} (G_0 G_1 G_2)^{2/3} (H_0 H_1 H_2)^{1/3} U_0^{2/3} (U_1 U_2)^{1/3}$$
$$\times \left(\max_{\{i,j,k\}=\{0,1,2\}} F_j F_k G_j^2 G_k^2 H_i^2 U_i \right)^{-2}.$$

Now, let us use the inequality

$$\left(\max_{\{i,j,k\}=\{0,1,2\}} F_j F_k G_j^2 G_k^2 H_i^2 U_i \right)^2 \geq F_0 (F_1 F_2)^{3/2} G_0^2 (G_1 G_2)^3 H_0^2 H_1 H_2 U_0 (U_1 U_2)^{1/2}.$$

This gives us

$$\mathcal{N}_2(B) \ll B(\log B) \sum_{\substack{F_i, G_i, H_i, U_i \\ i \in \{0,1,2\}}} (F_0 (F_1 F_2)^4 G_0^8 (G_1 G_2)^{14} H_0^{10} (H_1 H_2)^4 U_0^2 U_1 U_2)^{-1/6},$$

and therefore, we obtain

$$\mathcal{N}_2(B) \ll B \log B. \tag{4.5}$$

Putting together the three upper bounds (4.2), (4.4), and (4.5) completes the proof of the upper bound in Theorem 1.

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References

[1] Browning, T. D. "The density of rational points on a certain singular cubic surface." *Journal of Number Theory* 119, no. 2 (2006): 242–83.

[2] Browning, T. D. "The divisor problem for binary cubic forms." *Journal de Théorie des Nombres de Bordeaux* 23, no. 3 (2011): 579–602.

[3] Browning, T. D. and D. R. Heath-Brown. "Counting rational points on hypersurfaces." *Journal für die Reine und Angewandte Mathematik* 584 (2005): 83–115.

[4] Franke, J., Y. I. Manin, and Y. Tschinkel. "Rational points of bounded height on Fano varieties." *Inventiones Mathematicae* 95, no. 2 (1989): 421–35.

[5] Heath-Brown, D. R. "Diophantine approximation with square-free numbers." *Mathematische Zeitschrift* 187, no. 3 (1984): 335–44.

[6] Le Boudec, P. "Linear growth for certain elliptic fibrations." *International Mathematics Research Notices* 2015, no. 21 (2015): 10859-71.

[7] Peyre, E. "Hauteurs et mesures de Tamagawa sur les variétés de Fano." *Duke Mathematical Journal* 79, no. 1 (1995): 101–218.

[8] Robbiani, M. "On the number of rational points of bounded height on smooth bilinear hypersurfaces in biprojective space." *Journal of the London Mathematical Society* (2) 63, no. 1 (2001): 33–51.

[9] Schindler, D. "Bihomogeneous forms in many variables." *J. Théor. Nombres Bordeaux* (2013): preprint. arXiv:1301.6516v1.

[10] Schindler, D. "Manin's conjecture for certain biprojective hypersurfaces." *Journal für die Reine und Angewandte Mathematik* (2013): preprint. arXiv:1307.7069v1.

[11] Spencer, C. V. "The Manin conjecture for $x_0y_0 + \cdots + x_sy_s = 0$." *Journal of Number Theory* 129, no. 6 (2009): 1505–21.

[12] Thunder, J. L. "Asymptotic estimates for rational points of bounded height on flag varieties." *Compositio Mathematica* 88, no. 2 (1993): 155–86.