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Asymptotics for Helmholtz and Maxwell Solutions in 3-D Open Waveguides

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Abstract. We extend classic Sommerfeld and Silver-Müller radiation conditions for bounded scatterers to acoustic and electromagnetic fields propagating over three isotropic homogeneous layers in three dimensions. If $\mathbf{x}=(x_1,x_2,x_3)\in\mathbb{R}^3$, with x_3 denoting the direction orthogonal to the layers, standard conditions only hold for the outer layers in the region $|x_3|>\|\mathbf{x}\|^\gamma$, for $\gamma\in(1/4,1/2)$ and $\mathbf{x}$ large. For $|x_3|<\|\mathbf{x}\|^\gamma$ and inside the slab, asymptotic behavior depends on the presence of surface or guided modes given by the discrete spectrum of the associated operator.

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1 Introduction

Existence and uniqueness of acoustic and electromagnetic (EM) waves over layered structures have for long remained unsolved problems. These so-called *open waveguides* possess solutions that are divided, according to the continuous and discrete parts of the operator spectrum, into radiative and guided modes, respectively [12]. Guided or surface modes decay differently at infinity than radiative modes, and consequently, standard radiation conditions do not suffice to guarantee uniqueness. In [4], this is overcome by introducing a modal condition on the volume of a 2-D rectangular waveguide with varying coefficients in the core. A different approach is presented in [3] wherein one of the outer layers is replaced by a Dirichlet condition and uniqueness is achieved via a generalized Fourier transform.

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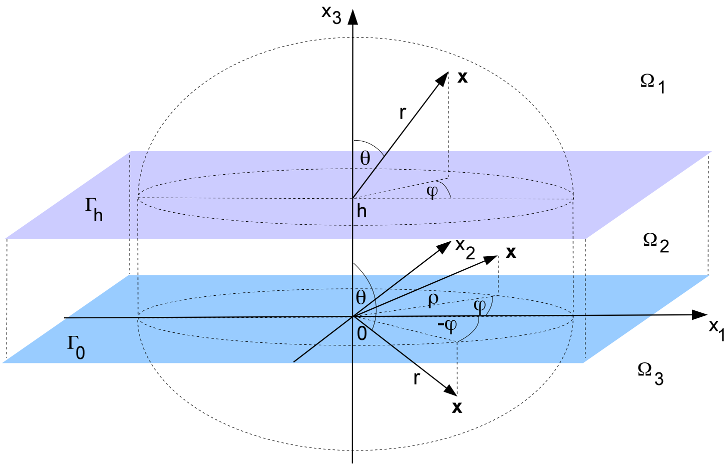

Figure 1: Coordinate and domain definitions.

In this work, we present rigorous asymptotics for outgoing acoustic and Maxwell waves in the time harmonic regime in $\mathbb{R}^3$ using the *limiting absorption principle* [9]. This constitutes a steppingstone towards a general existence result for open waveguides and uniqueness proofs in the fashion of [5]. On the application side, these precise asymptotic characterizations allow for the improvement of non-reflecting boundary conditions and perfectly matched layers (PML) based techniques in layered media [11].

1.1 Problem setting

1.1.1 Geometry and physical parameters

Let $h\in\mathbb{R}_+$, and define intervals $I_1:=(h,+\infty)$, $I_2:=(0,h)$ and $I_3:=(-\infty,0)$. We consider the following three-layer decomposition of $\mathbb{R}^3$ (see Fig. 1):

$$\Omega_1:=\{\mathbf{x}\in\mathbb{R}^3:x_3\in I_1\},\qquad \Omega_2:=\{\mathbf{x}\in\mathbb{R}^3:x_3\in I_2\},\qquad \Omega_3:=\{\mathbf{x}\in\mathbb{R}^3:x_3\in I_3\},$$

with interfaces $\Gamma_0:=\overline{\Omega}_2\cap\overline{\Omega}_3$, $\Gamma_h:=\overline{\Omega}_2\cap\overline{\Omega}_1$, and $\Omega:=\bigcup_i\Omega_i$. Introduce hemispherical coordinates (r,θ,φ) with origin at $(0,0,h)$ for Ω_1 and at $(0,0,0)$ for Ω_3. That is, for $r\in\mathbb{R}_+$, $\varphi\in(0,2\pi)$, and either $\theta\in(0,\frac{\pi}{2})$ in Ω_1 or $\theta\in(\frac{\pi}{2},\pi)$ in Ω_3, we have the equivalences:

$$x_1=r\sin\theta\cos\varphi,\quad x_2=r\sin\theta\sin\varphi,\quad \text{and}\quad x_3=\begin{cases}h+\cos\theta, & x_3\in I_1,\\ \cos\theta, & x_3\in I_3.\end{cases}$$

In Ω_2, we employ cylindrical coordinates (ρ,φ,x_3) with $\rho>0$, $\varphi\in(0,2\pi)$ and $x_3\in I_2$, so that $x_1=\rho\cos\varphi$ and $x_2=\rho\sin\varphi$.

Each domain Ω_i, $i=1,2,3$, is characterized by different parameters according to the physical situation considered. For linear electromagnetism, relative permittivity and permeability coefficients, $\epsilon_i,\mu_i\in L^\infty(\Omega_i)$, are both real and positive. Inside Ω_i, the light speed

c_i is equal to $c_0/\sqrt{\epsilon_i\mu_i}$ where c_0 is the speed of light in vacuum. In the acoustic case, real positive and bounded constants c_i refer to sound speeds. Parameters $\eta_i \in \mathbb{R}_+$, representing viscosities in acoustics or conductivities in electromagnetism, immediately guarantee the well-posedness of the system, i.e. bounded energy. Nonetheless, we will be mostly interested in the case when they tend to zero, and thus we set $\eta_i \equiv \eta$ in all layers, for simplicity.

1.1.2 Time-harmonic or Helmholtz formulation

Let $\omega \in \mathbb{R}_+$ and U_η be such that $\mathfrak{Re}\{U_\eta(\mathbf{x})e^{-\imath\omega t}\}$ represents either the space-time dependent scalar pressure field or the normal EM field-component describing transverse electric (TE) or transverse magnetic (TM) polarization. Specifically, if $(\mathbf{E},\mathbf{H}) \in \mathbb{C}^6$ denotes the pair of electromagnetic complex-valued vector fields, then $U_\eta = H_3$ and $U_\eta = E_3$ for the TE and TM case, respectively. The subscript η will be extensively used to emphasize the dependence on this parameter and is omitted whenever $\eta = 0$.

Define real and complex wavenumbers $k_i^2 := (\omega/c_i)^2$ and $k_{i,\eta}^2 := k_i^2 + \imath\omega\eta$. We are interested in solving the family of time-harmonic problems for η going to zero: find $U_\eta \in H^1_{\mathrm{loc}}(\Delta,\Omega)$, the space of local H^1-functions with ΔU_η in $L^2_{\mathrm{loc}}(\Omega)$, such that

$$(P_\eta) := \begin{cases} \Delta U_\eta(\mathbf{x}) + k_{i,\eta}^2 U_\eta(\mathbf{x}) = 0, & \mathbf{x} \in \Omega_i, \quad i = 1,3, \\ \Delta U_\eta(\mathbf{x}) + k_{2,\eta}^2 U_\eta = \mathsf{F}_U(\mathbf{x},\omega)F(\mathbf{x}), & \mathbf{x} \in \Omega_2, \\ [\alpha U_\eta] = 0, & \mathbf{x} \in \Gamma_0 \cup \Gamma_h, \\ [\partial_3 U_\eta] = 0, & \mathbf{x} \in \Gamma_0 \cup \Gamma_h, \end{cases} \tag{1.1}$$

where $[\cdot]$ denotes jumps across a boundary with α being equal to μ for TE, ϵ for TM, or one if acoustics are considered. The source term $F(\mathbf{x})$ is complex-valued and compactly supported in Ω_2[†]. The term F_U is a partial differential operator whose form varies for TE and TM polarizations, accordingly, but is equal to identity for sound scattering. More precisely, the electromagnetic source is given by a vector electric current $\boldsymbol{\mathcal{F}}$ which yields, for each polarization, three different scalar sources $\mathsf{F}^j_U\boldsymbol{\mathcal{F}} := (\mathsf{F}_{\mathcal{U}}\boldsymbol{\mathcal{F}})\cdot\hat{\mathbf{x}}_j$, $j = 1,2,3$ [7].

If η is strictly positive, the above problems are well-defined only in the space $H^1(\Delta,\Omega)$ which is strictly contained in $H^1_{\mathrm{loc}}(\Delta,\Omega)$. For $\eta \equiv 0$, this does not hold and $H^1(\Delta,\Omega)$ is no longer a suitable solution space. Henceforth, we assume the following:

Hypothesis 1.1. The real wavenumbers satisfy $0 < k_3 \le k_1 < k_2 < +\infty$.

When η vanishes, the limit problem $(P_0) := \lim_{\eta\downarrow 0}(P_\eta)$ presents a countable number of guided modes, whose wave-vectors are given by $\|\boldsymbol{\xi}\| = \xi_p^m$, with $\boldsymbol{\xi} \in \mathbb{R}^2$ and wavenumbers $\xi_p^m > 0$ for $m \in \mathbb{N}$ [7]. Actually, these modes are the poles in the surface spectral Green's function (see Section 2.1).

[†]If the source lies in the exterior layers only minor modifications need to be introduced as done in [7].

1.2 Main results: far-field asymptotics for (P_0)

Proposition 1.1 (Sommerfeld-type conditions)**.** Assume the existence of $M\in\mathbb{N}$ guided modes. Moreover, let us admit for the limit problem (P_0) the decomposition:

$$U=U_g+U_{\rm rad} \qquad \text{with} \qquad U_g=\sum_{m=1}^{M}\alpha_m U_p^m, \qquad \alpha_m\in\mathbb{C}, \tag{1.2}$$

where $U_{\rm rad}$ and U_g denote radiative and guided parts, respectively, the latter composed of allowed modes U_p^m related to each wavenumber ξ_p^m. Then, using the coordinate system previously introduced, for $\gamma\in(\frac{1}{4},\frac{1}{2})$ it holds

$$\begin{cases} \left|\frac{\partial U}{\partial r}-\imath k_i U\right|=\mathcal{O}\left(r^{-\left(2\gamma+\frac{1}{2}\right)}\right), & \mathbf{x}\in\Omega_i, \quad i=1,3, \quad |x_3|>r^\gamma,\\ \left|\frac{\partial U}{\partial r}-\imath\sum_{m=1}^{M}\alpha_m\xi_p^m U_p^m\right|=\mathcal{O}\left(r^{-\left(\frac{3}{2}-\gamma\right)}\right), & \mathbf{x}\in\Omega_i, \quad i=1,3, \quad |x_3|<r^\gamma,\\ \left|\frac{\partial U}{\partial \rho}-\imath\sum_{m=1}^{M}\alpha_m\xi_p^m U_p^m\right|=\mathcal{O}\left(\rho^{-\frac{3}{2}}\right), & \mathbf{x}\in\Omega_2. \end{cases} \tag{1.3}$$

The same conditions are satisfied by each component of the EM vector fields for both TE and TM polarizations.

Proposition 1.2 (Silver-Müller-type conditions)**.** Define the radiative and modal impedances $\mathsf{z}_{i,r}:=(\mu_i/\epsilon_i)^{1/2}$ and $\mathsf{z}_{i,\rho}^m:=\mathsf{z}_{i,r}\xi_p^m/k_i$. If excited by a compactly supported electrical current, outgoing transverse magnetic fields satisfy the following conditions:

$$\begin{cases} \left|\mathbf{H}^i+\mathsf{z}_{i,r}^{-1}\mathbf{E}\wedge\mathbf{n}\right|=\mathcal{O}\left(r^{-\left(2\gamma+\frac{1}{2}\right)}\right), & \mathbf{x}\in\Omega_i, \quad i=1,3, \quad |x_3|>r^\gamma,\\ \left|\mathbf{H}+\sum_{m=1}^{M}\alpha_m\left(\mathsf{z}_{i,\rho}^m\right)^{-1}\mathbf{E}_p^m\wedge\mathbf{n}\right|=\mathcal{O}\left(r^{-\left(\frac{3}{2}-\gamma\right)}\right), & \mathbf{x}\in\Omega_i, \quad i=1,3, \quad |x_3|<r^\gamma,\\ \left|\mathbf{H}+\sum_{m=1}^{M}\alpha_m\left(\mathsf{z}_{2,\rho}^m\right)^{-1}\mathbf{E}_p^m\wedge\mathbf{n}\right|=\mathcal{O}\left(\rho^{-\frac{3}{2}}\right), & \mathbf{x}\in\Omega_2, \end{cases} \tag{1.4}$$

where $\mathbf{E}_p^m$ represent the associated guided electric field modes described in Proposition 1.1, $\mathbf{n}=\mathbf{x}/r$ in $\mathbf{x}\in\Omega_i$, $i=1,3$, and $\mathbf{n}=\boldsymbol{\rho}/\rho$ in Ω_2. Similar conditions for transverse electric modes hold by reversing the roles of $\mathbf{H}$ and $\mathbf{E}$ and replacing $\mathsf{z}_{i,r}$, $\mathsf{z}_{i,\rho}^m$ with their inverses.

In what follows, we only provide the proof of Proposition 1.1 for the scalar form of the Helmholtz equation. Thus, henceforth we set in (1.1) $\alpha_i\equiv 1$, $i=1,2,3$ and $\mathsf{F}_U=\mathsf{Id}$. Extensions to the vector Helmholtz equation and Silver-Müller conditions are discussed in [7,8].

2 Asymptotics for the scalar Helmholtz problem

One can recover U_η for an arbitrary but compactly supported source by convolution with the associated Green's function, g_η, i.e. $U_\eta=g_\eta*F$. Hence, the far-field behavior is indeed the one induced by g_η. These distributions are obtained by replacing the source $F(\mathbf{x})$ with a delta Dirac distribution, δ, located at $\mathbf{y}\in\Omega_2$. By translational invariance, we can set $y_1=y_2=0$ so that the source's position is given only by $y_3\in I_2$. Application of the Fourier transform along the horizontal plane yields a system of ordinary differential equations (ODEs) in x_3 whose solution is given in Proposition 2.1. With this, in Section 2.2, we carry out the asymptotic analysis of the inverse surface Fourier transform when η goes to zero.

2.1 Surface spectral Green's function

Let $\mathbf{x}'=(x_1,x_2)$ and $\boldsymbol{\xi}=(\xi_1,\xi_2)$ belong to $\mathbb{R}^2$ with $\xi_1=\xi\cos\phi$ and $\xi_2=\xi\sin\phi$ for $\xi\in\mathbb{R}_+$ and $\phi\in(0,2\pi)$. With the unit vector $\mathbf{t}(\phi):=(\cos\phi,\sin\phi)$, the surface Fourier transform can be written as

$$f(\mathbf{x}',x_3)=\frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \widehat{f}(\xi,\phi,x_3)e^{-\imath\xi\mathbf{t}(\phi)\cdot\mathbf{x}'}\xi d\xi d\phi \tag{2.1}$$

with obvious inverse form. Define restrictions of the Green's function over each layer $g_\eta^i:=g_\eta|_{\Omega_i}$. After applying the Fourier transform to (1.1) one must solve the following problem: seek $\widehat{g}_\eta^i$, in the space of tempered distributions $S'(\mathbb{R}^2_{\boldsymbol{\xi}}\times I_i\times I_2)$, for $i=1,2,3$, such that, for $y_3\in I_2$, it holds

$$(\widehat{P}_\eta):=\begin{cases}\partial_3^2\widehat{g}_\eta^i-\chi_{i,\eta}^2\widehat{g}_\eta^i=0, & x_3\in I_i,\quad i=1,3,\\ \partial_3^2\widehat{g}_\eta^2-\chi_{2,\eta}^2\widehat{g}_\eta^2=\frac{1}{2\pi}\delta(x_3-y_3), & x_3\in I_2,\\ \left[\widehat{g}_\eta\right]=0, & x_3=0,h,\\ \left[\partial_3\widehat{g}_\eta\right]=0, & x_3=0,h,\\ +\text{decay conditions} & |x_3|\longrightarrow+\infty,\end{cases} \tag{2.2}$$

where $\chi_{i,\eta}^2(\xi):=\xi^2-k_{i,\eta}^2$. Solutions of the homogeneous ODEs take the form:

$$\widehat{g}_\eta^i(\boldsymbol{\xi},x_3,y_3)=K_{1,\eta}^i(\boldsymbol{\xi},y_3)e^{-(x_3-y_3)\chi_{i,\eta}}+K_{2,\eta}^i(\boldsymbol{\xi},y_3)e^{(x_3-y_3)\chi_{i,\eta}},\qquad i=1,3, \tag{2.3}$$

with distributions $K_{j,\eta}^i\in S'(\mathbb{R}^2_{\boldsymbol{\xi}}\times I_2)$, $j=1,2$, obtained by imposing boundary and decay conditions. We have defined the square root in the complex plane

$$\chi_{i,\eta}:z\longmapsto\sqrt{z^2-k_{i,\eta}^2},\qquad i=1,2,3, \tag{2.4}$$

as the product between $\sqrt{z-k_{i,\eta}}$ and $\sqrt{z+k_{i,\eta}}$, for $z\in\mathbb{C}$, $\mathfrak{Re}\{z\}=\xi$, such that

$$\arg(z-k_{i,\eta})\in(-3\pi/2,\pi/2)\qquad\text{and}\qquad\arg(z+k_{i,\eta})\in(-\pi/2,3\pi/2).$$

Remark 2.1. Set $\eta=0$. Then, if $\mathfrak{Im}\{z\}=0$ and $\mathfrak{Re}\{z\}=\xi\in\mathbb{R}$, it holds $\arg(\xi-k_i)\in\{-\pi,0\}$ and $\arg(\xi+k_i)\in\{0,\pi\}$ and thus, χ_i takes either real or purely complex values. The latter occurs if $|\xi|<k_i$.

Due to the punctual support of the exciting term, we introduce an artificial layer at $x_3=y_3$ and split the interval I_2 into $I_{2+}:=(y_3,h)$ and $I_{2-}:=(0,y_3)$. This induces a decomposition of the spatial domain Ω_2 into

$$\Omega_2^+=\{\mathbf{x}\in\Omega_2: x_3\in I_{2+}\} \qquad \text{and} \qquad \Omega_2^-=\{\mathbf{x}\in\Omega_2: x_3\in I_{2-}\},$$

so that $\widehat{g}_\eta^2$ is built by the solutions of two homogeneous ODEs for $x_3\in I_{2-}$ and $x_3\in I_{2+}$, linked by transmission conditions at $x_3=y_3$ [8]. For a wave traveling in Ω_i towards Ω_j, we introduce complex reflection and transmission coefficients, denoted by R_{ij}^η and T_{ij}^η, respectively, given by

$$R_{ij}^\eta(\xi):=\frac{\chi_{i,\eta}-\chi_{j,\eta}}{\chi_{j,\eta}+\chi_{i,\eta}}, \qquad T_{ij}^\eta(\xi):=1+R_{ij}^\eta=\frac{2\chi_{i,\eta}}{\chi_{j,\eta}+\chi_{i,\eta}}. \tag{2.5}$$

Observe that $R_{ji}^\eta=-R_{ij}^\eta$. Finally, we introduce the following complex-valued surface spectral function:

$$\mathrm{Det}_\eta(\xi):=R_{21}^\eta R_{23}^\eta \exp\left(-2h\chi_{2,\eta}\right)-1, \tag{2.6}$$

whose zeros represent the so-called *in-phase condition* necessary for surface modes (see [7] and references therein).

Proposition 2.1 ([8]). Introduce the excitation related term $\widehat{L}_\eta(\xi):=(4\pi\chi_{2,\eta})^{-1}$. If Det_η is non-zero, the solution to the spectral problem $(\widehat{P}_\eta)$ is

$$\widehat{g}_\eta^1(x_3)=\widehat{L}_\eta\frac{T_{21}^\eta}{\mathrm{Det}_\eta}e^{-(h-y_3)\chi_{2,\eta}}\left[1+R_{23}^\eta e^{-2y_3\chi_{2,\eta}}\right]e^{-(x_3-h)\chi_{1,\eta}}, \qquad x_3\in I_1, \tag{2.7a}$$

$$\widehat{g}_\eta^{2+}(x_3)=\frac{\widehat{L}_\eta}{\mathrm{Det}_\eta}\left[e^{y_3\chi_{2,\eta}}+R_{23}^\eta e^{-y_3\chi_{2,\eta}}\right]\left[e^{-x_3\chi_{2,\eta}}+R_{21}^\eta e^{-2h\chi_{2,\eta}}e^{x_3\chi_{2,\eta}}\right], \qquad x_3\in I_{2+}, \tag{2.7b}$$

$$\widehat{g}_\eta^{2-}(x_3)=\frac{\widehat{L}_\eta}{\mathrm{Det}_\eta}\left[e^{-y_3\chi_{2,\eta}}+R_{21}^\eta e^{-(2h-y_3)\chi_{2,\eta}}\right]\left[R_{23}^\eta e^{-x_3\chi_{2,\eta}}+e^{x_3\chi_{2,\eta}}\right], \qquad x_3\in I_{2-}, \tag{2.7c}$$

$$\widehat{g}_\eta^3(x_3)=\widehat{L}_\eta\frac{T_{23}^\eta}{\mathrm{Det}_\eta}\left[e^{-y_3\chi_{2,\eta}}+R_{21}^\eta e^{-(2h-y_3)\chi_{2,\eta}}\right]e^{x_3\chi_{3,\eta}}, \qquad x_3\in I_3, \tag{2.7d}$$

where the dependence on $(\boldsymbol{\xi},y_3)\in\mathbb{R}^2\times I_2$ is implied.

Remark 2.2. One can further simplify (2.7a)-(2.7d) into the general form:

$$\widehat{g}_\eta^i(\boldsymbol{\xi},x_3,y_3)=\widehat{L}_\eta(\xi)\frac{\Xi_\eta^i(\xi,y_3)}{\mathrm{Det}_\eta(\xi)}\times\text{exponential terms in } x_3, \tag{2.8}$$

where $\xi=|\boldsymbol{\xi}|$. Functions $\Xi_\eta^i(\xi,y_3)$ are built from terms R_{ij}^η,T_{ij}^η and $\chi_{i,\eta}$ which depend solely on the radial spectral coordinate.

2.2 Asymptotic analysis for vanishing absorption

We now compute asymptotics when η goes to zero for the inverse Fourier transforms:

$$g^i_\eta(\mathbf{x}',x_3,y_3) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \widehat{g}^i_\eta(\boldsymbol{\xi},x_3,y_3)e^{-\imath\xi\mathbf{t}(\phi)\cdot\mathbf{x}'}\xi d\xi d\phi, \tag{2.9}$$

with $\mathbf{x}'=(x_1,x_2)$, $\mathbf{y}=(0,0,y_3)$, and where $\widehat{g}^i_\eta$ has the form (2.8). For this, we rewrite the integrals (2.9) in the standard form:

$$g^i_\eta(\lambda,\cdot,\cdot,y_3) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \Psi^i_\eta(\xi,\phi,y_3)e^{\lambda\Phi^i_\eta(\xi,\phi,\cdot,\cdot)}\xi d\xi d\phi, \tag{2.10}$$

where $(\lambda,\cdot,\cdot)$ equals (r,θ,φ) or (ρ,φ,x_3) according to Ω_i with λ as the large parameter. The terms Φ^i_η denote the associated phase, obtained by multiplying exponential terms coming from $\widehat{g}^i_\eta$ and the Fourier transform exponential. The remaining terms are bundled up in the amplitude function Ψ^i_η.

2.2.1 General procedure

Without loss of generality, let us assume the existence of only two opposed poles at $\pm\xi_p\in\mathbb{R}$ with $\xi_p>0$ and when $\eta=0$. For small $\eta>0$, the real poles $|\xi_p|$ are displaced as $\xi_{p,\eta}\sim\pm(\xi_p+\imath\eta)$ [6]. The complete asymptotic behavior on each layer, denoted by $\mathbb{I}^i$, is obtained by letting η go to zero and adding the following contributions:

1. $\mathbb{I}^i_{\boldsymbol{\xi}^i_s}$, due to stationary points $\boldsymbol{\xi}^i_s$ in the phase function Φ^i and derived via the *stationary phase method* [10]; and,

2. $\mathbb{I}^i_{\xi^i_c}$, defined as

$$\mathbb{I}^i_{\xi^i_c}(\lambda,\cdot,\cdot,y_3) = \frac{1}{2\pi}\int_0^{2\pi} J^i_{\xi^i_c}(\lambda,\cdot,\cdot,y_3,\phi)d\phi, \tag{2.11}$$

due to critical points $\boldsymbol{\xi}^i_c$ in the radial integral J^i. Specifically, for a fixed integration angle ϕ, we study

$$J^i(\lambda,\cdot,\cdot,y_3,\phi) := \lim_{\eta\downarrow 0}\int_0^\infty \Psi^i_\eta(\xi,\phi,y_3)e^{\lambda\Phi^i_\eta(\xi,\phi,\cdot,\cdot)}\xi d\xi, \quad \lambda\to+\infty, \tag{2.12}$$

by replacing ξ with the complex variable z and using the *residue theorem* [1] for the complex contours shown in Fig. 2 for $\mathfrak{Re}\{z\}\geq 0$. This requires analytic continuations for Ψ^i_η, Φ^i_η based on the one for $\chi_{i,\eta}$ (2.4). Possible critical complex (real) points z_c (ξ_c) associated to the integral in z are

- surface mode or pole contributions at $z_c=\pm\xi_{p,\eta}$, given by the complex residue;
- branch points located at $z_c=\pm k_{i,\eta}$ for $i=1,2,3$; and
- integration end-points at $z_c=0\pm\imath\eta$,

whose asymptotics are found via the *steepest descent method* [2]. After taking the limit $\eta\downarrow 0$, these last results are finally integrated with respect to ϕ.

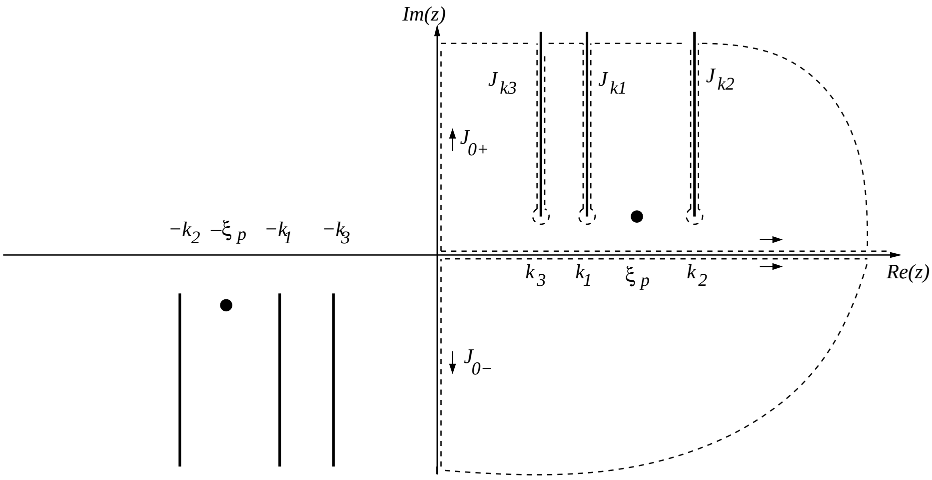

Figure 2: Complex paths for integrals in z at a fixed spectral angle ϕ (Initial form).

2.2.2 Explicit integrand forms

Recall the different coordinate systems presented in Section 1.1.1. For Ω_i, $i=1,3$, the amplitudes and phases in (2.10) are given by

$$\Psi^i_\eta(\xi,\phi,y_3) := \widehat{L}_\eta(\xi)\frac{\Xi^i_\eta(\xi,y_3)}{\mathrm{Det}_\eta(\xi)}, \tag{2.13a}$$

$$\Phi^i_\eta(\xi,\phi,\theta,\varphi) := -|\cos\theta|\chi_{i,\eta}(\xi) - \imath\xi\sin\theta\cos(\phi-\varphi). \tag{2.13b}$$

The phase expression yields both stationary points and branch points as shown below. In Ω_2, asymptotics are obtained along horizontal directions for

$$g^{2\pm}(\rho,\varphi,x_3^\pm,y_3) = \lim_{\eta\downarrow 0}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty}\Psi^{2\pm}_\eta(\xi,\varphi,y_3)e^{\rho\Phi^2(\xi,\phi,\varphi)}\xi d\xi d\phi, \tag{2.14}$$

where now

$$\begin{aligned}
&\Psi^{2\pm}_\eta(\xi,x_3^\pm,y_3) := \widehat{L}_\eta(\xi)\frac{\Xi^{2\pm}_\eta(\xi,y_3)}{\mathrm{Det}_\eta(\xi)}X^{2\pm}_\eta(\xi,x_3^\pm),\\
&\Phi^2(\xi,\phi,\varphi) := -\imath\xi\cos(\phi-\varphi),\\
&X^{2+}_\eta(\xi,x_3^+) := e^{-x_3^+\chi_{2,\eta}} + R^\eta_{21}e^{-2h\chi_{2,\eta}}e^{x_3^+\chi_{2,\eta}},\\
&X^{2-}_\eta(\xi,x_3^-) := R^\eta_{23}e^{-x_3^-\chi_{2,\eta}} + e^{x_3^-\chi_{2,\eta}},
\end{aligned}$$

with functions $X^{2\pm}_\eta$ and $\Xi^{2\pm}_\eta$ well-defined in ξ. Given the form of Φ^2, it is clear that it does not depend on η and that no saddle points occur for the integral in ξ. Thus, the only asymptotic behavior of $g^{2\pm}$ is given by the pole contribution.

2.2.3 Stationary point contributions for Ω_i, $i=1,3$

Use the form (2.9), let $\eta\equiv 0$, and multiply the integrand by a cut-off function $\vartheta\in\mathcal{D}(\mathbb{R}^2)$ such that ϑ is equal to one on a neighborhood of the stationary point $\boldsymbol{\xi}_s^i$ and zero elsewhere. Let $B_k(\boldsymbol{\xi}_s^i)\subset\mathbb{R}^2$ denote the ball of radius $k\in\mathbb{R}_+$ centered at $\boldsymbol{\xi}_s^i$. We calculate

$$\mathbb{I}^i_{\boldsymbol{\xi}_s^i}(r,\theta,\varphi,y_3)\sim\frac{1}{2\pi}\int_{B_k(\boldsymbol{\xi}_s)}\Psi^i(\xi_1,\xi_2,y_3)e^{\imath r\tilde{\Phi}^i(\xi_1,\xi_2,\theta,\varphi)}d\xi_1 d\xi_2,$$

wherein we have followed Remark 2.1 to modify the phase (2.13b) by defining

$$\tilde{\Phi}^i(\xi_1,\xi_2,\theta,\varphi):=|\cos\theta|\sqrt{k_i^2-\xi^2}-\xi_1\sin\theta\cos\varphi-\xi_2\sin\theta\sin\varphi.$$

The only stationary point is

$$\boldsymbol{\xi}_s^i=(\xi_s^i,\xi_s^i)=(-k_i\sin\theta\cos\varphi,-k_i\sin\theta\sin\varphi),\qquad i=1,3,$$

which lies in the ball $B_{k_i}(\mathbf{0})=\{\boldsymbol{\xi}:|\boldsymbol{\xi}|\leq k_i\}$ with Hessian matrix:

$$\mathrm{H}_{\tilde{\Phi}^i}(\boldsymbol{\xi}_s^i)=-\frac{1}{k_i}\begin{pmatrix}1+\tan^2\theta\cos^2\varphi & \tan^2\theta\cos\varphi\sin\varphi\\ \tan^2\theta\cos\varphi\sin\varphi & 1+\tan^2\theta\sin^2\varphi\end{pmatrix}$$

satisfying $\det\mathrm{H}_{\tilde{\Phi}^i}(\boldsymbol{\xi}_s^i)=\sec^2\theta/k_i^2$. Moreover, the matrix has eigenvalues of opposite signs and consequently the signature of $\mathrm{H}_{\tilde{\Phi}^i}(\boldsymbol{\xi}_s^i)$ is zero, the stationary point thus being a saddle point. Application of the stationary phase method yields

$$\mathbb{I}^i_{\boldsymbol{\xi}_s^i}(r,\theta,\varphi,y_3)=k_i|\cos\theta|\Psi^i(\boldsymbol{\xi}_s^i,y_3)\frac{e^{\imath k_i r}}{r}+\mathcal{O}(r^{-2}),\qquad i=1,3. \tag{2.16}$$

Remark 2.3. If $\sin\theta=0$, the stationary point is also a critical point for J^i [see (2.12)], i.e. the end-point at $z=0$, and the above result is divided by two [10].

2.2.4 Surface mode or pole contribution for Ω_i, $i=1,3$

We now consider asymptotic contributions along a fixed angle J^i (2.12). We choose the complex paths so as to eliminate the integral contributions for large z and apply Jordan's lemma (see Fig. 2), i.e.

$$J^i_{\xi_p}=\imath 2\pi\lim_{\eta\downarrow 0}\mathrm{Res}_{z=\xi_{p,\eta}}\left(z\Psi^i_\eta e^{r\Phi^i_\eta}\right). \tag{2.17}$$

By looking at our square root definition (2.4), we write $z=Re^{\imath\tau}$ with $R\in\mathbb{R}$, $\tau\in(-\frac{\pi}{2},\frac{\pi}{2})$ and study the integrand. First, we analyze the ubiquitous $\chi_{i,\eta}$:

$$\lim_{|R|\to+\infty}\chi_{i,\eta}=\lim_{|R|\to+\infty}\sqrt{(Re^{\imath\tau})^2-k_{i,\eta}^2}=\lim_{|R|\to+\infty}|R|e^{\imath\tau},\quad i=1,2,3, \tag{2.18}$$

and consequently, all terms T_{ij}^{η}, R_{ij}^{η} are bounded. Second, observe that exponential terms $\exp(\pm s\chi_{i,\eta})$, with $s\in\mathbb{R}_+$, behave as

$$\lim_{|R|\to+\infty}\exp(\pm s\chi_{i,\eta})=\lim_{|R|\to+\infty}\exp(\pm s|R|e^{\imath\tau})=\lim_{|R|\to+\infty}\exp[\pm s|R|\cos\tau],$$

which, in view of the chosen domain for τ, converge to zero only for the negative sign. Hence, from (2.6),

$$\lim_{|R|\to+\infty}\mathrm{Det}_{\eta}(Re^{\imath\tau})=1.$$

Since the particular expressions for Ξ_{η}^{i} in (2.8) are well-defined and bounded, Ψ_{η}^{i} decays as $1/R$ for large z due to $\widehat{L}_{\eta}$. Finally, consider the real part of the exponential term:

$$\mathfrak{Re}\left\{e^{-r|\cos\theta||R|\exp(\imath\tau)}e^{-\imath|R|\exp(\imath\tau)r\sin\theta\cos(\phi-\varphi)}\right\}=e^{-r|R|(|\cos\theta|\cos\tau-\sin\tau\sin\theta\cos(\phi-\varphi))}.$$

For both Ω_1 and Ω_3, the elevation angle θ lies in $(0,\pi)$, and therefore $\sin\theta$ and $|\cos\theta|$ are positive. Thus, we define the integration contours in relation exclusively to the sign of $\cos(\phi-\varphi)$ so that integrals over paths at a fixed distance R vanish as R goes to infinity.

Case $\cos(\phi-\varphi)\geq 0$. Path integrals lie on the lower half-plane following the sense shown in Fig. 2. Hence, poles are not included and the only potential contribution comes from the integral departing from $z=0$:

$$J_{0^-}^{i}(r,\theta,\varphi,y_3,\phi)\sim\int_{0^+-\imath 0^+}^{0^+-\imath\infty}\Psi_{\eta}^{i}(z,\phi,y_3)e^{r\Phi_{\eta}^{i}(z,\phi,\theta,\varphi)}zdz\quad \lambda\to+\infty,\quad \eta\downarrow 0,\quad i=1,3,$$

lying on the fourth quadrant of the complex plane. The contribution is calculated in Section 2.2.6.

Case $\cos(\phi-\varphi)\leq 0$. Integrate over $\mathbb{C}_+$ and encircle the pole located at $\xi_{p,\eta}$. Its residue for vanishing η is

$$\begin{aligned}\lim_{\eta\downarrow 0}\mathrm{Res}_{z=\xi_{p,\eta}}\left(z\Psi_{\eta}^{i}e^{r\Phi_{\eta}^{i}}\right)&=\lim_{\eta\downarrow 0}\lim_{z\to\xi_{p,\eta}}(z-\xi_{p,\eta})z\Psi_{\eta}^{i}(z,\phi,y_3)e^{r\Phi^{i}(z,\phi,\theta,\varphi)}\\&=\xi_p\widehat{L}(\xi_p)\Xi^{i}(\xi_p,y_3)e^{r\Phi^{i}(\xi_p,\theta,\varphi,\phi)}\lim_{z\to\xi_p}\frac{z-\xi_p}{\mathrm{Det}(z)},\end{aligned}\tag{2.19}$$

wherein we have exchange limits by analyticity over the cut complex plane and functions $\widehat{L}$, Ξ^i and Φ^i [see (2.13a)] are well-defined at ξ_p. Since the determinant is null when $\eta\equiv 0$, we take the last limit using l'Hôpital's rule:

$$\lim_{z\to\xi_p}\frac{z-\xi_p}{\mathrm{Det}(z)}=\lim_{z\to\xi_p}\left[\mathrm{Det}'(z)\right]^{-1}.$$

The derivative of the determinant can be found as follows: let $f(z) = \mathrm{Det}(z)+1$, take logarithm and derive in z. Evaluation at $z=\xi_p$ yields $f(\xi_p)=1$, and consequently,

$$\mathrm{Det}'(\xi_p) = \frac{R'_{21}(\xi_p)}{R_{21}(\xi_p)} + \frac{R'_{23}(\xi_p)}{R_{23}(\xi_p)} - 2h\chi'_2(\xi_p).$$

Observe that $\chi'_i = \xi/\chi_i$, and therefore,

$$\mathrm{Det}'(\xi_p) = 2\xi_p\chi_2^{-1}(\xi_p)\widetilde{\mathrm{Det}}'(\xi_p)$$

with

$$\widetilde{\mathrm{Det}}'(\xi_p) := -\chi_1^{-1}(\xi_p) - \chi_3^{-1}(\xi_p) - h.$$

Since $k_{1,3} \neq k_2$ and $h>0$, the above quantity is well-defined and one can conclude

$$J^i_{\xi_p}(r,\theta,\varphi,y_3,\phi) = \imath 2\pi\chi_2(\xi_p)\widehat{L}(\xi_p)\frac{\Xi^i(\xi_p,y_3)}{2\widetilde{\mathrm{Det}}'(\xi_p)}e^{-r|\cos\theta|\sqrt{\xi_p^2-k_i^2}-\imath r\xi_p\sin\theta\cos(\phi-\varphi)}, \quad i=1,3$$

whenever $\cos(\phi-\varphi) \leq 0$.

Angular integration. From (2.11), we write

$$\mathbb{I}^i_{\xi_p}(r,\theta,\varphi,y_3) = \imath\chi_2(\xi_p)\widehat{L}(\xi_p)\frac{\Xi^i(\xi_p,y_3)}{2\widetilde{\mathrm{Det}}'(\xi_p)}e^{-r|\cos\theta|\sqrt{\xi_p^2-k_i^2}}W^i_{\xi_p}(r\sin\theta,\varphi),$$

for $i=1,3$, where the last term is obtained as follows. Since the residue is zero for $\cos(\phi-\varphi)>0$, we use the indicator function $\mathbf{1}_A(\varphi)$ equal to one when $\varphi\in A$ and zero elsewhere, so that

$$W^i_{\xi_p}(\rho,\varphi) := \int_0^{2\pi}\mathbf{1}_{\{\phi:\cos(\phi-\varphi)<0\}}(\phi)e^{-\imath\xi_p\rho\cos(\phi-\varphi)}d\phi,$$

with ρ being the projection of r over the equatorial plane, i.e. $\rho = r\sin\theta$. Application of the stationary phase method, for the phase rewritten as $w(\phi) := -\xi_p\cos(\phi-\varphi)$, yields stationary points at $\sin(\phi-\varphi)=0$, i.e. $\phi^s = m\pi+\varphi$, with $m=0,1,2$ at most. This gives

$$w(\phi^s) = -\xi_p(-1)^m, \qquad \partial^2_\phi w(\phi^s) = \xi_p(-1)^m.$$

However, $\mathbf{1}_{\{\phi:\cos(\phi-\varphi)<0\}}$ is nonzero only for $m=1$. Thus, bearing in mind that both ϕ and φ belong to the interval $(0,2\pi)$, the method yields

$$W^i_{\xi_p}(\rho,\varphi) = \left(\frac{2\pi}{\rho\xi_p}\right)^{1/2}e^{\imath\rho\xi_p-\imath\pi/4} + \mathcal{O}\left(\rho^{-3/2}\right). \tag{2.20}$$

Summarizing results for $i=1,3$, we obtain‡

$$\mathbb{I}^i_{\xi_p} = \chi_2(\xi_p)\widehat{L}(\xi_p)\frac{\Xi^i(\xi_p,y_3)}{2\widetilde{\mathrm{Det}}'(\xi_p)}e^{-r|\cos\theta|\sqrt{\xi_p^2-k_i^2}}\left(\frac{2\pi}{\rho\xi_p}\right)^{1/2}e^{\imath\rho\xi_p+\imath\pi/4} + \mathcal{O}\left(\rho^{-3/2}\right),$$

where the phase $-\imath\pi/4$ is changed due to the $\imath$ factor coming from the residue theorem.

‡The surface pole contribution for Ω_2 is identically computed to the one above [8] but it only depends on ρ.

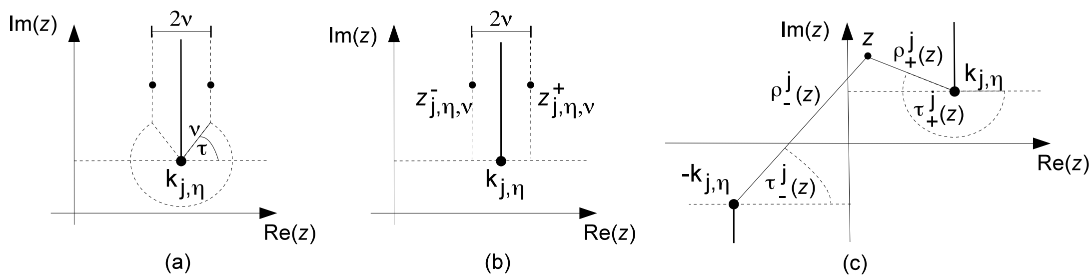

Figure 3: Local integration coordinates for branch cut integrals.

Remark 2.4. The function decreases exponentially in the vertical direction, whereas the decrease is as $\rho^{-1/2}$ as we approach the $x_3=\{0,h\}$ planes. If the function Det possesses many zeros, each associated residue must be added.

2.2.5 Branch point contributions

From Fig. 2, relevant branch points start from $k_{j,\eta}$, j=1,2,3, yielding three potential contributions $J^i_{k_j}$, for i=1,3, when η vanishes. At each branch cut, the original contour follows a loop-hole (see Fig. 3). First, we show that the integrals are well-defined around these points and therefore integral paths can be as close as desired to the branch cut.

Loop-hole integrals. At $k_{j,\eta}$, we use the local coordinates described in Fig. 3(a) and calculate the limits:

$$\lim_{\nu\downarrow 0}\left|\Psi^i_\eta(k_{j,\eta}+\nu e^{\imath\tau},\phi,y_3)e^{r\Phi^i_\eta(k_{j,\eta}+\nu e^{\imath\tau},\phi,\theta,\varphi)}\right|,\quad j=1,2,3,\quad i=1,3.$$

Clearly, for $j\neq i$, $\lim_{\nu\downarrow 0}\chi_{i,\eta}(k_{j,\eta}+\nu e^{\imath\tau})=\sqrt{k^2_{j,\eta}-k^2_{i,\eta}}$ is well-defined. When $i=j$, we have

$$\lim_{\nu\downarrow 0}\sqrt{2k_{i,\eta}+\nu e^{\imath\tau}}\sqrt{\nu e^{\imath\tau}}=\sqrt{2k_{i,\eta}}e^{\imath\tau/2}\lim_{\nu\downarrow 0}\sqrt{\nu} \tag{2.21}$$

and, consequently, coefficients R^η_{ij} and T^η_{ij} are bounded. Thus, functions Ξ^i_η and Det_η are well-behaved around points $k_{j,\eta}$ for all $j=1,2,3$. On the other hand, the source $\widehat{L}_\eta$ is singular as $\nu^{-1/2}$ when $z\to k_{2,\eta}$ [see (2.21)]. Since the Jacobian is equal to ν around $k_{j,\eta}$ for all cases, integrals

$$\lim_{\nu\downarrow 0}\int_{-3\pi/2}^{\pi/2}\Psi^i_\eta(k_{j,\eta}+\nu e^{\imath\tau},\phi,y_3)e^{r\Phi^i_\eta(k_{i,\eta}+\nu e^{\imath\tau},\phi,\theta,\varphi)}(k_{i,\eta}+\nu e^{\imath\tau})\nu e^{\imath\tau}d\tau$$

vanish. Thus, we only need to compute vertical integrals on either side of the branch cuts.

Integrals parallel to the branch cuts. Introduce $z_{j,\eta} := \imath s + k_{j,\eta}$ and $z^{\pm}_{j,\eta,\nu} := z_{j,\eta} \pm \nu$ with $s \in \mathbb{R}_+$ as the new integration variable [Fig. 3(b)]. We must compute

$$J^i_{k_j} = \lim_{\eta,\nu\downarrow 0}\left(\int_{+\infty}^{0}\Psi^i_\eta(z^-_{j,\eta,\nu})e^{r\Phi^i_\eta(z^-_{j,\eta,\nu})}z^-_{j,\eta,\nu}dz^-_{j,\eta,\nu} + \int_0^{+\infty}\Psi^i_\eta(z^+_{j,\eta,\nu})e^{r\Phi^i_\eta(z^+_{j,\eta,\nu})}z^+_{j,\eta,\nu}dz^+_{j,\eta,\nu}\right). \tag{2.22}$$

For simplicity, introduce local polar coordinates shown in Fig. 3(c):

$$\begin{pmatrix}\rho^j_\pm(z)\\ \tau^j_\pm(z)\end{pmatrix} := \begin{pmatrix}|z\mp k_{j,\eta}|\\ \arg(z\mp k_{j,\eta})\end{pmatrix}, \quad \rho^j_\pm \in \mathbb{R}_+, \quad \tau^j_+ \in \left(-\frac{3\pi}{2},\frac{\pi}{2}\right), \quad \tau^j_- \in \left(-\frac{\pi}{2},\frac{3\pi}{2}\right), \tag{2.23}$$

for $z \in \mathbb{C}$ and $j=1,2,3$. When ν goes to zero, the angle τ^j_+ takes the values

$$\tau^j_+\left(z^+_{j,\eta,0}\right) = \frac{\pi}{2} \quad \text{and} \quad \tau^j_+\left(z^-_{j,\eta,0}\right) = -\frac{3\pi}{2}, \qquad s\in\mathbb{R}_+,$$

while τ^j_- does not vary. Notice that coordinates $(\rho^j_\pm, \tau^j_\pm)$ remain unchanged if defined with respect to $z^\pm_{i,\eta,0}$ with $i \neq j$. Based on the representation

$$\chi_{i,\eta}(z) = \sqrt{\rho^i_+\rho^i_-}\exp\left[\imath\left(\frac{\tau_+ + \tau_-}{2}\right)\right],$$

we can state the following relations:

$$\begin{aligned} &\chi_i(z^+_{i,\eta,0}) = \sqrt{\rho^i_+\rho^i_-}e^{\imath\pi/4}e^{\imath\tau^i_-/2}, && \chi_i(z^-_{i,\eta,0}) = -\chi_i(z^+_{i,\eta,0}),\\ &\chi_j(z^+_{i,\eta,0}) = \chi_j(z^-_{i,\eta,0}), && i\neq j, \end{aligned} \tag{2.24}$$

and deduce

$$R^\eta_{ij}(z^-_{i,\eta,0}) = \left[R^\eta_{ij}(z^+_{i,\eta,0})\right]^{-1} \quad \text{and} \quad T^\eta_{ij}(z^-_{i,\eta,0}) = T^\eta_{ij}(z^+_{i,\eta,0})\left[R^\eta_{ij}(z^+_{i,\eta,0})\right]^{-1}.$$

Consequently,

$$\mathsf{Det}(z^-_{i,\eta,0}) = -\mathsf{Det}(z^+_{i,\eta,0})\left[R_{21}(z^+_{i,\eta,0})R_{23}(z^+_{i,\eta,0})e^{-2h\chi_2(z^+_{i,\eta,0})}\right]^{-1}.$$

With the above, the reader can verify the helpful result:

$$\Psi^i_\eta(z^-_{j,\eta,0}) = \Psi^i_\eta(z^+_{j,\eta,0}) = \Psi^i_\eta(z_{j,\eta}), \qquad j=1,2,3, \quad i=1,3,$$

and one can write integrals (2.22) as

$$J^i_{k_j} = \lim_{\eta\downarrow 0}\lim_{R\to\infty}\int_0^R \Psi^i_\eta(z_{j,\eta})\left[e^{r\Phi^i_\eta(z^+_{j,\eta,0})} - e^{r\Phi^i_\eta(z^-_{j,\eta,0})}\right]z_{j,\eta}(s)dz_{j,\eta}(s). \tag{2.25}$$

Clearly, the phase Φ^i_η does not change at either side of the branch cuts located at $k_{j,\eta}$ for $j\neq i$ [see (2.13b)] due to property (2.24). Thus, the square brackets term in (2.25) is equal to zero, and

$$\mathbb{I}^i_{k_j}(r,\theta,\varphi)=0, \qquad j\neq i, \quad i=1,3.$$

Contribution when $i=j$. On the other hand, Φ^i_η does change when crossing the branch cut located in $k_{i,\eta}$ as it passes through the Riemann sheets of $\chi_{i,\eta}$. Replacing Φ^i_η in (2.25), yields

$$J^i_{k_i}=\lim_{\substack{\eta\downarrow 0\\ R\to\infty}}\int_0^R \Psi^i_\eta(z_{i,\eta})e^{\imath r z_{i,\eta}\sin\theta|\cos(\phi-\varphi)|}\left[e^{-r|\cos\theta|\chi_{i,\eta}(z^+_{i,\eta,0})}-e^{r|\cos\theta|\chi_{i,\eta}(z^+_{i,\eta,0})}\right]z_{i,\eta}(s)\,dz_{i,\eta}(s).$$

In the following, we deform the original contour from Fig. 2 to that given by the steepest descent direction whilst taking the limit in η. For $0\leq\theta<\pi/2$ and $\cos(\phi-\varphi)<0$, we study the phase when z approaches $k_{i,\eta}$:

$$\Phi^i(z)\sim \imath k_i\sin\theta|\cos(\phi-\varphi)|-|\cos\theta|\sqrt{2k_i}(z-k_i)^{1/2}.$$

By identifying the above with (A.5), we obtain $a=\cos\theta\sqrt{2k_i}$, $\alpha=\pi$ and $n=1/2$. From (A.2), the angle $\Theta_p=0$ and we modify the original contour so that the integral now goes along $\mathfrak{Re}\{z\}=0$. We then analyze the integral

$$J^i_{k_i}=\int_{C_{k_i}}\Psi^i(z)e^{\imath r z\sin\theta|\cos(\phi-\varphi)|}e^{-r\cos\theta\chi(z)}z\,dz,$$

where C_{k_i} is the steepest descent path for which the imaginary part of the phase is kept constant, i.e.

$$\mathfrak{Im}\left\{\Phi^i(z)-\Phi^i(k_i)\right\}=0. \tag{2.26}$$

Using the coordinates defined in (2.23) [see Fig. 3(c)] satisfying

$$\rho^i_+\sin\tau^i_+=\rho^i_-\sin\tau^i_-, \qquad \rho^i_+\cos\tau^i_++2k_i=\rho^i_-\cos\tau^i_-, \quad \text{for} \quad 0\leq\tau^i_-<\tau^i_+\leq\pi/2,$$

and express condition (2.26) as

$$\begin{aligned}&\mathfrak{Im}\left\{\imath\rho^i_+e^{\imath\tau^i_+}\sin\theta|\cos(\phi-\varphi)|-\cos\theta\sqrt{\rho^i_+\rho^i_-}\,e^{\imath(\tau_++\tau_-)/2}\right\}\sim 0,\\ &\rho^i_+\cos\tau^i_+\sin\theta|\cos(\phi-\varphi)|-\cos\theta\sqrt{\rho^i_+\rho^i_-}\sin\left(\frac{\tau_++\tau_-}{2}\right)\sim 0.\end{aligned}$$

In the first quadrant, for large $|z|$ it holds $\rho_-\sim\rho_+$ and $\tau_+\sim\tau_-$. Thus,

$$\tan\theta|\cos(\phi-\varphi)|\sim\tan\tau_+.$$

Although the steepest descent path depends upon $\tan\theta|\cos(\phi-\varphi)|$, it is always located on the first quadrant of the complex plane as $\theta\in(0,\pi/2)$ and

$$0\leq\tan\theta|\cos(\phi-\varphi)|\leq\tan\theta.$$

If $\theta=0$, τ_+ vanishes. This is consistent with a steepest descent path following the real axis when there is no oscillatory term in Φ^i. Thus, asymptotically, the path followed is that of a line with slope $\tan\tau_+$ whose main contribution is given by (A.8)

$$J^i_{k_i}(r,\theta,\phi,\varphi)\sim\Psi^i(k_i,\phi,y_3)\frac{1}{r^2\cos^2\theta}e^{\imath rk_i\sin\theta|\cos(\phi-\varphi)|}+\mathcal{O}(r^{-3}),\qquad\theta\in(0,\pi/2),$$

as $\widehat{L}(z,\phi)$ is well-defined at k_i for $i\neq 2$.

Angular integration. In the special case $\theta=\pi/2$, the term $\mathbb{I}^i_{k_i}$ vanishes. If $\theta\in(0,\pi/2)$, we apply the stationary phase method by using the same results provided in Section 2.2.4, i.e.

$$\mathbb{I}^i_{k_i}(r,\theta,\varphi)=\frac{1}{2\pi}\frac{1}{r^2\cos^2\theta}\frac{\Xi^i(k_i,y_3)}{\mathrm{Det}(k_i)}\widehat{L}(k_i)\left(\frac{2\pi}{\rho k_i}\right)^{1/2}e^{\imath\rho k_i-\imath\pi/4}+\mathcal{O}\left(\rho^{-3/2}\right).\tag{2.27}$$

2.2.6 End point contributions

Consider the integrals departing from $z=0$ towards $\pm\imath\infty$ shown in Fig. 2:

$$J^i_{0^\pm}=\lim_{\eta,\nu\downarrow 0}\int_0^\infty\Psi^i_\eta(z^\pm_{0,\nu},\phi,y_3)e^{r\Phi^i_\eta(\theta,\varphi,z^\pm_{0,\nu},\phi)}z^\pm_{0,\nu}(s)dz^\pm_{0,\nu}(s),$$

where $z^\pm_{0,\nu}:=\pm\imath s+\nu$ with $s,\nu\in\mathbb{R}_+$. We study the phase at $s=0$ for the integral in z using the derivative

$$\partial_z\Phi^i_\eta=-|\cos\theta|z\chi^{-1}_{i,\eta}-\imath\sin\theta\cos(\phi-\varphi).\tag{2.28}$$

If $\theta>0$ or $\cos(\phi-\varphi)\neq 0$, the end point is neither a stationary point nor a branch point, and we can set $n=1$ and use formula (A.7) from the steepest descent method. Taking the limit in η, from (2.28), $\alpha=\mp\pi/2$ depending on the sign of $\cos(\phi-\varphi)$, $\Theta_1=\pi-\alpha$ and $|\partial_z\Phi^i(0)|=\sin\theta|\cos(\phi-\varphi)|$. Therefore, $\beta=2$ in (A.7) and the integrals in z for both signs of the cosine are asymptotically equal to

$$\begin{aligned}J^i_{0^\mp}(r,\theta,\varphi,y_3,\phi)&=\widehat{L}(0)\frac{\Xi^i(0,y_3)}{\mathrm{Det}(0)}\frac{1}{(r\sin\theta|\cos(\phi-\varphi)|)^2}e^{\imath rk_i\cos\theta\pm\imath\pi/2}+\mathcal{O}\left(\rho^{-4}\right)\\&=\pm\widehat{L}(0)\frac{\Xi^i(0,y_3)}{\mathrm{Det}(0)}\frac{1}{\rho^2|\cos(\phi-\varphi)|^2}e^{\imath rk_1\cos\theta+\imath\pi/2}+\mathcal{O}\left(\rho^{-4}\right),\end{aligned}$$

the plus and minus signs coming from the phase in $\pi/2$.

Angular integration. Integration over ϕ yields,

$$\mathbb{I}^i_0(r,\theta,\varphi)=\frac{1}{2\pi}\int_0^{2\pi}\left(\mathbf{1}_{\{\phi:\cos(\phi-\varphi)<0\}}J^i_{0^+}(r,\phi,\varphi)+\mathbf{1}_{\{\phi:\cos(\phi-\varphi)>0\}}J^i_{0^-}(r,\phi,\varphi)\right)d\phi=0$$

since $\theta>0$ and $\cos(\phi-\varphi)\neq 0$, the denominators never vanish and the above integrals are bounded. Moreover, regardless of the sign of $\cos(\phi-\varphi)$ they have the same result so that contributions of order ρ^{-2} cancel each other.

2.2.7 Results for scalar Helmholtz and proof of Proposition 1.1

Proposition 2.2. Consider the coordinate sets describing for each Ω_i defined before. Assume the existence of a single surface mode ξ_p and let $\gamma\in(\frac{1}{4},\frac{1}{2})$. Then, the far-field of the scalar or vectorial Green's functions g^i when η vanishes is given by

- For Ω_i, $i=1,3$ and $r|\cos\theta|>r^\gamma$,

$$g^i(r,\theta,\varphi,y_3)=\Lambda_r^{U,i}(\theta,\varphi,y_3)\frac{e^{\imath k_i r}}{r}+\mathcal{O}\left(r^{-(2\gamma+\frac{1}{2})}\right)$$

  and, for $0<r|\cos\theta|<r^\gamma$:

$$g^i(r,\theta,\varphi,y_3)=\Lambda_\rho^{U,i}(\varphi,y_3)e^{-r\cos\theta\sqrt{\xi_p^2-k_i^2}}\frac{e^{\imath\rho\xi_p+\imath\pi/4}}{\rho^{1/2}}+\mathcal{O}\left(r^{-(\frac{3}{2}-\gamma)}\right)$$

- On the other hand, for $\Omega_{2\pm}$, we have

$$g^2(\rho,\varphi,x_3,y_3)=\Lambda_\rho^{U,2\pm}(\varphi,y_3)X^{2\pm}(\xi_p,x_3)\frac{e^{\imath\rho\xi_p+\imath\pi/4}}{\rho^{1/2}}+\mathcal{O}\left(\rho^{-3/2}\right) \tag{2.29}$$

with according scalar or vector terms depending on the precise field U described

$$\begin{aligned}
\Lambda_r^{U,i}(\theta,\varphi,y_3)&:=k_i|\cos\theta|\widehat{L}^U(-k_i\sin\theta,\varphi)\frac{\Xi^i(k_i\sin\theta,y_3)}{\mathrm{Det}(k_i\sin\theta)}, \quad i=1,3,\\
\Lambda_\rho^{U,i}(\varphi,y_3)&:=-\chi_2(\xi_p)\widehat{L}^U(\xi_p,\varphi)\frac{\Xi^i(\xi_p,y_3)}{2\widetilde{\mathrm{Det}}'(\xi_p)}\left(\frac{2\pi}{\xi_p}\right)^{1/2}, \quad i=1,2\pm,3.
\end{aligned}$$

Proof of Proposition 1.1. Fields U are built via the volume integral between the source F and the derived Green's functions $g^i(\mathbf{x},\mathbf{y})$, $i=1,2,3$, with fixed $\mathbf{x}$. Since F is compactly supported, we can take a point $\mathbf{x}_0$ on the exterior of the integration domain and write the integral kernel as $g^i(\mathbf{x},\mathbf{y})-g^i(\mathbf{x}_0,\mathbf{y})+g^i(\mathbf{x}_0,\mathbf{y})$. One notices that the asymptotics do not depend on y_3. Then, one can easily show that the term $g^i(\mathbf{x},\mathbf{y})-g^i(\mathbf{x}_0,\mathbf{y})$ decays much faster than $g^i(\mathbf{x}_0,\mathbf{y})$ for large $\mathbf{x}_0$ outside the support, and thus, the asymptotic behavior of the solution is precisely given by that of $g^i(\mathbf{x}_0,\mathbf{y})$ (cf. Proposition 2.2). □

Remark 2.5. In the vectorial Helmholtz and EM cases, the proofs of Propositions 1.1 and 1.2 are carried out almost identically as shown in [7,8].

3 Conclusion and extensions

We have extended radiation conditions for compactly supported excitations in layered isotropic media. This allows the construction of suitable bases for both theoretical and numerical use. Furthermore, one can extend these results via the same methodology to more layers or excitations outside the guide. However, the existence of modal decompositions is crucial for the conditions to hold.

Appendix

We summarize a few results derived via the method of the steepest descents on the asymptotics of the complex integral

$$I(\lambda) \sim \int_C g(z) e^{\lambda\Phi(z)} dz. \tag{A.1}$$

Theorem A.1 ([1,2])**.** *Let all derivatives up to order $n-1$ of Φ vanish at a point z_0, i.e.*

$$\frac{d^q\Phi}{dz^q}\Big|_{z=z_0} = 0, \quad q=1,\cdots,n-1, \quad \textit{and} \quad \frac{1}{n!}\frac{d^n\Phi}{dz^n}\Big|_{z=z_0} = ae^{\imath\alpha}, \quad a>0.$$

If $z-z_0=\rho e^{\imath\theta}$, then the directions of steepest descent are given by

$$\Theta_p = -\frac{\alpha}{n} + (2p+1)\frac{\pi}{n}, \quad p=0,\cdots,n-1. \tag{A.2}$$

A generalization of the above for non-integer n is obtained by setting:

$$\Phi(z) \sim \Phi(z_0) + ae^{\imath\alpha}(z-z_0)^n$$

in some sector of the z-plane with apex in z_0. Then, the directions of steepest descent at z_0 are also given by (A.2). Several cases can occur

- *Saddle point at regular point of $g(z)$:*

$$I(\lambda) \sim \frac{g(z_0)}{n}\left[\frac{n!}{\lambda\left|\Phi^{(n)}(z_0)\right|}\right]^{1/n} \Gamma\left(\frac{1}{n}\right) e^{\lambda\Phi(z_0)+\imath\Theta_p}. \tag{A.3}$$

- *Saddle point in $\Phi(z)$ and branch point in $g(z)$:* we write $g(z) \sim g_0(z-z_0)^{\beta-1}$, $z\to z_0$ yielding

$$I(\lambda) \sim \frac{g_0}{n}\left[\frac{n!}{\lambda\left|\Phi^{(n)}(z_0)\right|}\right]^{\beta/n} \Gamma\left(\frac{\beta}{n}\right) e^{\lambda\Phi(z_0)+\imath\beta\Theta_p}. \tag{A.4}$$

- *Branch point in both* $\Phi(z)$ *and* $g(z)$: we write

$$\Phi(z) \sim \Phi(z_0) + ae^{\imath\alpha}(z-z_0)^n, \quad n \in \mathbb{R}, \tag{A.5}$$

$$I(\lambda) \sim \frac{g_0}{n} \frac{1}{(\lambda a)^{\beta/n}} \Gamma\left(\frac{\beta}{n}\right) e^{\lambda\Phi(z_0)+\imath\beta\Theta_p}. \tag{A.6}$$

- *Only a branch point in* $g(z)$ *and* $n=1$: we write, $\Theta_1 = \pi - \alpha$, and

$$I(\lambda) \sim \frac{g_0}{(\lambda|\Phi'(z_0)|)^\beta} \Gamma(\beta) e^{\lambda\Phi(z_0)+\imath\beta\Theta_1}. \tag{A.7}$$

- *Branch point only in* $\Phi(z)$:

$$I(r) \sim \frac{g(z_0)}{n} \frac{1}{(\lambda a)^{1/n}} \Gamma\left(\frac{1}{n}\right) e^{\lambda\Phi(z_0)+\imath\Theta_p}. \tag{A.8}$$

References

[1] M. Ablowitz and A. Fokas. Complex Variables: Introduction and Applications. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, UK, 2nd edition, 2003.

[2] N. Bleistein. Mathematical Methods for Wave Phenomena. Computer Science and Applied Mathematics. Academic Press, Orlando, USA, 1984.

[3] A.-S. Bonnet-Ben Dhia, G. Dakhia, C. Hazard, and L. Chorfi. Diffraction by a defect in an open waveguide: a mathematical analysis based on a modal radiation condition. SIAM J. Appl. Math., 70(3):677–693, 2009.

[4] G. Ciraolo and R. Magnanini. A radiation condition for uniqueness in a wave propagation problem for 2-D open waveguides. Math. Methods Appl. Sci., 32(10):1183–1206, 2009.

[5] M. Durán, I. Muga, and J.-C. Nédélec. The Helmholtz equation in a locally perturbed half-space with non-absorbing boundary. Archive for Rational Mechanics and Analysis, 191(1):143–172, 2009.

[6] J. G. Harris. Linear Elastic Waves. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, UK, second edition, 2001.

[7] C. Jerez-Hanckes. Modeling Elastic and Electromagnetic Surface Waves in Piezoelectric Tranducers and Optical Waveguides. PhD thesis, École Polytechnique, Palaiseau, France, 2008.

[8] C. Jerez-Hanckes and J. C. Nédélec. Asymptotics for Helmholtz and Maxwell solutions in 3-D open waveguides. Technical report, Seminar for Applied Mathematics, ETH Zurich, 2010.

[9] R. Leis. Initial Boundary Value Problems in Mathematical Physics. B.G. Teubner, Stuttgart, 1986.

[10] J. Murray. Asymptotic Analysis. Number 48 in Applied Mathematical Sciences. Springer-Verlag, Inc., New York, USA, 1984.

[11] F. Olyslager. Discretization of continuous spectra based on perfectly matched layers. SIAM J. Appl. Math., 64(4):1408–1433, 2004.

[12] R. Weder. Spectral and scattering theory for wave propagation in perturbed stratified media. Number 87 in Applied Mathematical Sciences. Springer-Verlag, Inc., New York, USA, 1981.