Published in "Numerische Mathematik 2012, doi: 10.1007/s00211-011-0442-8"
which should be cited to refer to this work.

On the Lebesgue constant of barycentric rational interpolation at equidistant nodes

Len Bos* Stefano De Marchi† Kai Hormann‡ Georges Klein§

Abstract

Recent results reveal that the family of barycentric rational interpolants introduced by Floater and Hormann is very well-suited for the approximation of functions as well as their derivatives, integrals and primitives. Especially in the case of equidistant interpolation nodes, these infinitely smooth interpolants offer a much better choice than their polynomial analogue. A natural and important question concerns the condition of this rational approximation method. In this paper we extend a recent study of the Lebesgue function and constant associated with Berrut's rational interpolant at equidistant nodes to the family of Floater–Hormann interpolants, which includes the former as a special case.

Math Subject Classification: 65D05, 65F35, 41A05, 41A20

Keywords: Barycentric rational interpolation, Lebesgue function, condition

1 Introduction

The approximation problem we consider is the following: suppose we want to approximate a function $f\colon [a,b] \to \mathbb{R}$ by some g, taken from a finite-dimensional linear subspace of the Banach space $C^0[a,b]$ of continuous functions over $[a,b]$ with the maximum norm, such that g interpolates f at the $n+1$ distinct interpolation nodes $a = x_0 < x_1 < \cdots < x_n = b$,

$$g(x_k) = f(x_k), \qquad k = 0,\dots,n.$$

With a given set of *basis functions* b_j satisfying the *Lagrange property*,

$$b_j(x_k) = \delta_{jk} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k, \end{cases}$$

we may define the class of interpolants that we focus on, namely that of linear interpolants

$$g(x) = \sum_{j=0}^{n} b_j(x) f(x_j).$$

We stress that by linearity we mean the dependence on the data $f(x_0),\dots,f(x_n)$. Examples include (among many others) polynomial interpolation in Lagrangian form and linear barycentric rational interpolation. Besides the convergence theory of g, it is natural to study the condition of this numerical approximation method.

*Department of Computer Science, University of Verona, Strada le Grazie 15, 37134 Verona, Italy, *leonard-peter.bos@univr.it*

†Department of Pure and Applied Mathematics, University of Padova, Via Trieste 63, 35121 Padova, Italy, *demarchi@math.unipd.it*

‡Faculty of Informatics, University of Lugano, Via Giuseppe Buffi 13, 6904 Lugano, Switzerland, *kai.hormann@usi.ch*

§Department of Mathematics, University of Fribourg, Pérolles, 1700 Fribourg, Switzerland, *georges.klein@unifr.ch*

Let every data point $f(x_j)$ be given with an absolute error or perturbation of at most ε, for example due to rounding, noise, or measurement imprecision. Then the maximum distance in $[a, b]$ between the interpolant $\tilde{g}$ of the perturbed data and the interpolant g of the exact data is bounded as

$$\max_{a \le x \le b} |\tilde{g}(x) - g(x)| \le \varepsilon \max_{a \le x \le b} \Lambda_n(x),$$

where the function

$$\Lambda_n(x) = \sum_{j=0}^{n} |b_j(x)|$$

is called the *Lebesgue function* for the nodes $x_0, x_1, \ldots, x_n$, and its maximum over the interval $[a, b]$,

$$\Lambda_n = \max_{a \le x \le b} \Lambda_n(x),$$

is the *Lebesgue constant* [14]. Thus, the quantity Λ_n is the worst possible error amplification and, since g is linear in the data, coincides with the condition number of the interpolation process [13]. Throughout this paper we make use of this interpretation of the Lebesgue constant. Its original definition as the norm of the approximation operator [14] is not needed here, since we are not looking for the *best* approximation of f in some linear space.

Numerous authors have derived results about the Lebesgue function and constant associated with Lagrange interpolation at various kinds of nodes; see [5, 6, 18] and the references therein. It is well known [5] that the Lebesgue constant associated with Lagrange interpolation at nodes distributed in any way always increases at least logarithmically with the number of nodes. Such a rate is achieved, for instance, for Chebyshev nodes [15, 18].

In contrast to this favourable behaviour, the Lebesgue constant for Lagrange interpolation at *equidistant nodes* grows exponentially,

$$\Lambda_n \sim \frac{2^{n+1}}{en \ln(n)}$$

as $n \to \infty$. More detailed results and other approaches to describing the error amplification may be found in [7, 10, 17, 19] and the references therein. The bad condition, together with Runge's phenomenon [8, 16], makes Lagrange interpolation at equidistant nodes often useless for $n \ge 50$. In fact, interpolation at these nodes is a challenging problem, as it was shown in [13] that it is not possible to develop an interpolation method which is simultaneously well-conditioned and converging at an exponential rate as the number of nodes increases. One way to overcome this restriction and to get better results is using rational instead of polynomial interpolation at equidistant nodes.

Berrut and Mittelmann [3] determine rational interpolants with small Lebesgue constants for given nodes by numerically optimizing the denominator of the interpolant. Here we shall concentrate on the family of barycentric rational interpolants introduced by Floater and Hormann [9] with basis functions

$$b_j(x) = \frac{(-1)^j \beta_j}{x - x_j} \Bigg/ \sum_{i=0}^{n} \frac{(-1)^i \beta_i}{x - x_i}, \qquad j = 0, \ldots, n. \tag{1}$$

Explicit formulas for the positive weights $\beta_0, \ldots, \beta_n$ are given in the same paper. The original construction reveals that the so-obtained rational interpolant is a blend of local polynomial interpolants of degree at most d corresponding to $d + 1$ consecutive values of the given function. It is further shown that the approximation order is $O(h^{d+1})$, where $h = \max_{0 \le i \le n-1}(x_{i+1} - x_i)$, as long as the interpolated function is $d + 2$ times continuously differentiable. This family of barycentric rational interpolants is well-suited for the approximation of sufficiently smooth functions [9] as well as for applications such as the approximation of derivatives, integrals and primitives [2, 11, 12].

For $n \ge 2d$ equidistant nodes, which is the setting that we assume from now on, the weights in (1) turn out to be

$$\beta_j = \sum_{k=d}^{n} \binom{d}{k-j} = \begin{cases} \sum_{k=0}^{j} \binom{d}{k}, & \text{if } j \le d, \\ 2^d, & \text{if } d \le j \le n - d, \\ \beta_{n-j}, & \text{if } j \ge n - d. \end{cases} \tag{2}$$

If $d = 0$, then all weights are equal to one, and the favourable properties of the corresponding rational interpolant were discovered numerically by Berrut [1]. Bos, De Marchi, and Hormann [4] later analysed the associated Lebesgue constant and show that it is bounded as

$$c_n \ln(n+1) \leq \Lambda_n \leq 2 + \ln(n), \tag{3}$$

where $c_n = 2n/(4+n\pi)$ with $\lim_{n\to\infty} c_n = 2/\pi$.

The general case $d \geq 1$ needs a different approach since the study of the Lebesgue function

$$\Lambda_n(x) = \sum_{j=0}^{n} |b_j(x)| = \sum_{j=0}^{n} \frac{\beta_j}{|x-x_j|} \Bigg/ \left| \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x-x_j} \right| \tag{4}$$

through the direct use of the basis functions (1) results in rather complicated expressions, whereas the original form of the rational interpolants as blends of polynomials allows for much shorter proofs.

The aim of this paper is to show that the Lebesgue constant associated with the family of Floater–Hormann interpolants with $d \geq 1$ grows logarithmically in the number of interpolation nodes if these are equidistant. This is achieved by establishing logarithmic upper and lower bounds in Sections 2 and 3, respectively.

2 Upper bound

In case of equidistant nodes, the properties of barycentric rational interpolation depend only on the constant distance h between the nodes. For simplicity and without loss of generality, we assume that the interpolation interval is $[0,1]$, so that the nodes are equally spaced with distance $h = 1/n$,

$$x_k = kh = \frac{k}{n}, \qquad k = 0, \ldots, n.$$

We begin by deriving an upper bound for the Lebesgue constant associated with the family of Floater–Hormann interpolants with $d \geq 1$.

Theorem 1. *The Lebesgue constant associated with rational interpolation at equidistant nodes with the basis functions $b_j(x)$ in* (1) *satisfies*

$$\Lambda_n \leq 2^{d-1}(2 + \ln n).$$

Proof. If $x = x_k$ for $k = 0, \ldots, n$, then $\Lambda_n(x) = 1$. Otherwise, $x_k < x < x_{k+1}$ for some k with $0 \leq k \leq n-1$ and we consider

$$\Lambda_n(x) = \frac{(x-x_k)(x_{k+1}-x) \displaystyle\sum_{j=0}^{n} \frac{\beta_j}{|x-x_j|}}{(x-x_k)(x_{k+1}-x) \left| \displaystyle\sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x-x_j} \right|} =: \frac{N_k(x)}{D_k(x)}.$$

Since all the weights β_j are less than or equal to 2^d, the numerator is bounded as

$$N_k(x) \leq 2^d \left(\frac{1}{n} + \frac{1}{2n} \ln n \right),$$

following the proof of Theorem 2 in [4] for the case $d = 0$.

We now show that the denominator $D_k(x)$ is bounded from below by $1/n$, which leads to the claimed result. To see this, we recall from [9, Section 4] that

$$\sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x-x_j} = (-1)^d d! h^d \sum_{j=0}^{n-d} \lambda_j(x), \tag{5}$$

where

$$\lambda_j(x) = \frac{(-1)^j}{(x - x_j)\cdots(x - x_{j+d})}.$$

Assuming $k \le n - d$, the proof of Theorem 2 in [9] shows that

$$\left|\sum_{j=0}^{n-d} \lambda_j(x)\right| \ge |\lambda_k(x)| = \frac{1}{(x - x_k)(x_{k+1} - x)\cdots(x_{k+d} - x)}. \tag{6}$$

Therefore,

$$\begin{aligned} D_k(x) &= d!h^d(x - x_k)(x_{k+1} - x)\left|\sum_{j=0}^{n-d} \lambda_j(x)\right| \\ &\ge \frac{d!h^d}{\prod_{j=k+2}^{k+d}(x_j - x)} \ge \frac{d!h^d}{\prod_{j=k+2}^{k+d}(x_j - x_k)} = h = \frac{1}{n}, \end{aligned}$$

where the last inequality follows from the fact that $x_j - x \le x_j - x_k$ for $j \ge k+1$. If $k > n-d$, a similar reasoning leads to this lower bound for $D_k(x)$ by considering $\lambda_{k-d+1}(x)$ instead of $\lambda_k(x)$ in (6). $\square$

Note that the upper bound in Theorem 1 for $d = 1$ is identical to the upper bound for $d = 0$ in (3), which is consistent with our numerical observations that both cases have a similar Lebesgue constant; compare Figure 3 (top right) and Figure 2 in [4].

3 Lower bound

Let us now turn to the study of the lower bound of the Lebesgue constant associated with the family of Floater–Hormann interpolants. We first give a general result for any $d \ge 1$ and then derive an improved bound for the case $d = 1$, which turns out to be again very similar to the one for $d = 0$ in (3).

Theorem 2. *The Lebesgue constant associated with rational interpolation at equidistant nodes with the basis functions $b_j(x)$ in (1) satisfies*

$$\Lambda_n \ge \frac{1}{2^{d+2}}\binom{2d+1}{d}\ln\left(\frac{n}{d} - 1\right).$$

Proof. From numerical experiments (see Figure 1), it appears that for $d \ge 2$ the Lebesgue function

$$\Lambda_n(x) = \frac{\displaystyle\sum_{j=0}^{n} \frac{\beta_j}{|x - x_j|}}{\displaystyle\left|\sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x - x_j}\right|} =: \frac{N(x)}{D(x)} \tag{7}$$

obtains its maximum approximately halfway between x_0 and x_1 (and halfway between x_{n-1} and x_n because of the symmetry with respect to the mid-point of the interval). For this reason, we consider $x^* = (x_1 - x_0)/2 = 1/(2n)$ and derive a lower bound for $\Lambda_n(x^*)$.

We begin by investigating the numerator at x^*,

$$N(x^*) = \sum_{j=0}^{n} \frac{\beta_j}{|x^* - x_j|} = \sum_{j=0}^{n} \frac{\beta_j}{\left|\frac{1}{2n} - \frac{j}{n}\right|} = 2n\sum_{j=0}^{n} \frac{\beta_j}{|2j - 1|}.$$

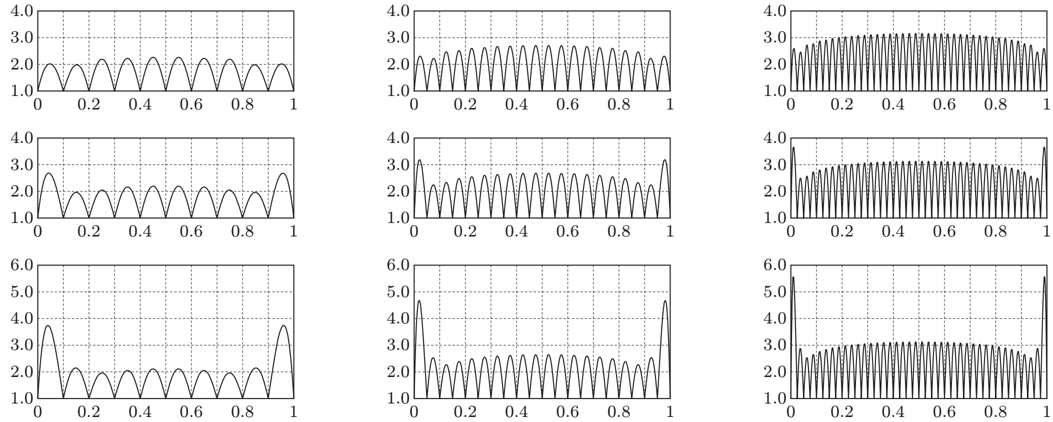

Figure 1: Lebesgue function of the Floater–Hormann interpolants for $d = 1$ (top), $d = 2$ (middle), and $d = 3$ (bottom) at $n + 1$ equidistant nodes for $n = 10, 20, 40$.

Omitting the first and last d terms and noticing that $\beta_j = 2^d$ for the remaining terms, we obtain

$$N(x^*) \geq 2n2^d \sum_{j=d}^{n-d} \frac{1}{2j-1} \geq 2n2^d \int_{d-1}^{n-d} \frac{1}{2x+1} dx = 2n2^d \frac{1}{2} \ln\left(\frac{2n-2d+1}{2d-1}\right)$$
$$\geq n2^d \ln\left(\frac{n}{d} - 1\right).$$

To handle the denominator, we first recall (5) to get

$$D(x^*) = \left| \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x^* - x_j} \right| = d! h^d \left| \sum_{j=0}^{n-d} \lambda_j(x^*) \right|.$$

As x^* belongs to the first sub-interval $[x_0, x_1]$, we notice that $\lambda_0(x^*)$ and $\lambda_1(x^*)$ have the same sign and that the following $\lambda_j(x^*)$ oscillate in sign and decrease in absolute value. The absolute value of the sum over these functions is thus bounded from above by the sum of the absolute values of the first two terms,

$$D(x^*) \leq d! h^d \big(|\lambda_0(x^*)| + |\lambda_1(x^*)|\big),$$

and expanding this expression at x^* finally gives

$$D(x^*) \leq d! h^d \left(\frac{1}{\prod_{j=0}^{d} \left|\frac{1}{2n} - \frac{j}{n}\right|} + \frac{1}{\prod_{j=1}^{d+1} \left|\frac{1}{2n} - \frac{j}{n}\right|} \right)$$
$$= n2^{d+1} d! \left(\frac{2d+1}{\prod_{j=1}^{d+1}(2j-1)} + \frac{1}{\prod_{j=1}^{d+1}(2j-1)} \right) = n \frac{2^{2d+2}}{\binom{2d+1}{d}}.$$

□

Proposition 1. *If $d = 1$, then the Lebesgue constant associated with rational interpolation at equidistant nodes with the basis functions $b_j(x)$ in (1) satisfies*

$$\Lambda_n \geq a_n \ln(n) + b_n,$$

where $\lim_{n\to\infty} a_n = 2/\pi$ *and* $\lim_{n\to\infty} b_n = 0$.

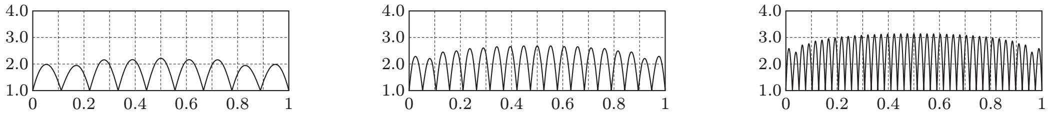

Figure 2: Lebesgue function of the Floater–Hormann interpolants for $d = 1$ at $n+1$ equidistant nodes for $n = 9, 19, 39$.

Proof. If $d = 1$, then the weights β_j in (2) simplify to

$$\beta_j = \begin{cases} 1, & \text{if } j = 0, \\ 2, & \text{if } 1 \le j \le n-1, \\ 1, & \text{if } j = n. \end{cases} \tag{8}$$

Assume first that $n = 2k+1$ is odd. The proof is very similar to that of Theorem 2, except that we use $x^* = 1/2$. According to our numerical experiments, this is where the maximum of the Lebesgue function appears to occur (see Figure 2).

We first derive a lower bound for the denominator $D(x^*)$. Due to the symmetry of $D(x)$ with respect to x^*, the first and the last $k+1$ terms in the sum are equal and so

$$\begin{aligned} D(x^*) &= \left| \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x^* - x_j} \right| = 2n \left| \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{n - 2j} \right| \\ &= 4n \left| \sum_{j=0}^{k} \frac{(-1)^j \beta_j}{2k+1-2j} \right| = 4n \left| \sum_{j=0}^{k} \frac{(-1)^j \beta_{k-j}}{2j+1} \right|. \end{aligned}$$

Now using the triangle inequality, Equation (8), and the fact that the Leibniz series converges to $\pi/4$ with

$$\left| \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} - \frac{\pi}{4} \right| \le \frac{1}{2k+1},$$

we have

$$\begin{aligned} D(x^*) &\le 4n \left(\left| \sum_{j=0}^{k-1} \frac{(-1)^j \beta_{k-j}}{2j+1} \right| + \left| \frac{(-1)^k \beta_0}{2k+1} \right| \right) = 8n \left| \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} \right| + \frac{4n}{2k+1} \\ &\le 8n \left(\frac{\pi}{4} + \frac{1}{2k+1} \right) + 4 = 2n\pi + 12. \end{aligned} \tag{9}$$

It remains to find a lower bound for the numerator $N(x^*)$. With the same arguments as above, it follows that

$$N(x^*) = 2n \sum_{j=0}^{n} \frac{\beta_j}{|n-2j|} = 4n \sum_{j=0}^{k} \frac{\beta_j}{2k+1-2j} = 8n \sum_{j=0}^{k-1} \frac{1}{2j+1} + \frac{4n}{2k+1} \ge 4n \ln(n) + 4,$$

and together with (9) we obtain

$$\Lambda_n \ge \frac{2}{\pi + 6/n} \ln(n) + \frac{2}{n\pi + 6}.$$

Finally, if $n = 2k$ is even, then the point $x = 1/2$ is a node and the Lebesgue function equals one there. Referring to Figure 1, we consider $x^* = 1/2 + 1/(2n)$ instead in this case. Applying the same reasoning as for odd n leads to

$$D(x^*) = 2n \left| 4 \sum_{j=0}^{k-1} \frac{(-1)^{k+j}}{2j+1} + \frac{1}{2k-1} + \frac{1}{2k+1} \right| \le 2n\pi + \frac{2n}{n-1} + \frac{10n}{n+1}$$

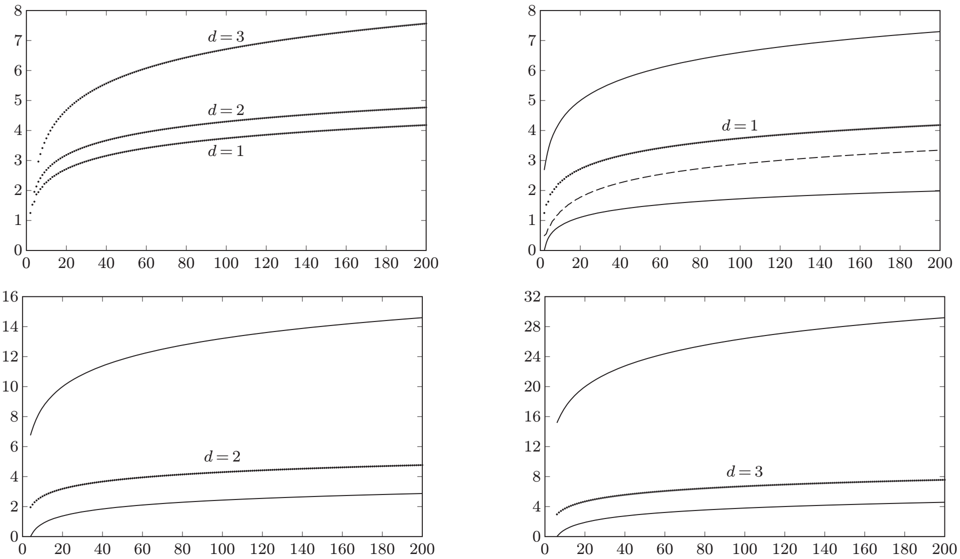

Figure 3: Comparison of the Lebesgue constants of the Floater–Hormann interpolants at $n+1$ equidistant nodes for $2d \leq n \leq 200$ and $d = 1, 2, 3$ (top left) and to the upper and lower bounds in Theorems 1 and 2. For $d = 1$, the improved lower bound in Proposition 1 is shown by the dashed curve (top right).

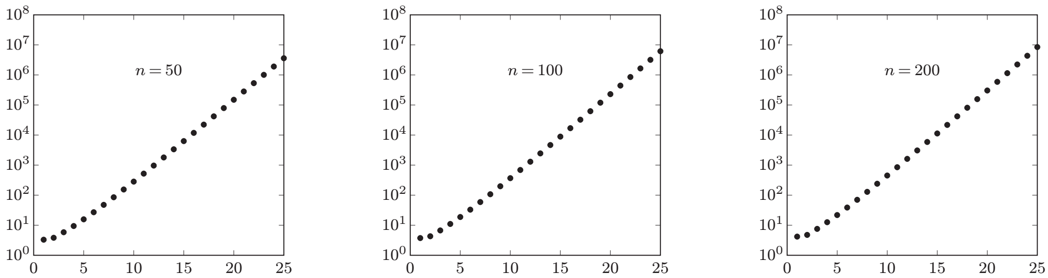

Figure 4: Lebesgue constants of the Floater–Hormann interpolants at a fixed number of $n+1$ equidistant nodes for $1 \leq d \leq 25$.

and

$$N(x^*) = 2n\left(4\sum_{j=0}^{k-1}\frac{1}{2j+1} - \frac{1}{2k-1} + \frac{1}{2k+1}\right) \geq 4n\ln(n+1) - \frac{2n}{n-1} + \frac{2n}{n+1},$$

hence

$$\Lambda_n \geq \frac{2}{\pi + \frac{6n-4}{n^2-1}}\ln(n+1) - \frac{2}{\pi(n^2-1)+6n-4},$$

which concludes the proof. $\square$

4 Numerical experiments

We performed numerous experiments to verify numerically that the behaviour of the Lebesgue constant associated with the family of barycentric rational interpolants is as predicted by the theoretical results in the previous sections. Figure 3 confirms that the growth of Λ_n is logarithmic in the number of interpolation nodes. These results further suggest that for fixed d the coefficient $\binom{2d+1}{d}/2^{d+2}$ of the logarithmic term in our lower bound in Theorem 2 is a better estimate of the true value than the factor

2^{d-1} in our upper bound in Theorem 1. However, both factors indicate that for fixed n the growth of the Lebesgue constant with respect to d is exponential, which is confirmed by the examples in Figure 4. Finally, Figure 1 demonstrates that this exponential growth seems to always happen near the boundary of the interpolation interval, whereas the behaviour of the Lebesgue function away from the boundary is almost independent of d. This suggests considering distributions of nodes which are uniform in the centre and clustered towards the boundary; we plan to study such settings in a forthcoming paper.

Acknowledgement: This work was carried out with support of the *60% funds* of the University of Padova and the corresponding funds of the University of Verona. The fourth author was partly supported by the Swiss National Science Foundation under grant No. 200020-124779. We would like to thank the reviewers for their valuable comments.

References

[1] J.-P. Berrut. Rational functions for guaranteed and experimentally well-conditioned global interpolation. *Comput. Math. Appl.*, 15(1):1–16, 1988.

[2] J.-P. Berrut, M. S. Floater, and G. Klein. Convergence rates of derivatives of a family of barycentric rational interpolants. *Appl. Numer. Math.*, 61(9):989–1000, Sept. 2011.

[3] J.-P. Berrut and H. D. Mittelmann. Lebesgue constant minimizing linear rational interpolation of continuous functions over the interval. *Comput. Math. Appl.*, 33(6):77–86, Mar. 1997.

[4] L. Bos, S. De Marchi, and K. Hormann. On the Lebesgue constant of Berrut's rational interpolant at equidistant nodes. *J. Comput. Appl. Math.*, 236(4):504–510, Sept. 2011.

[5] L. Brutman. On the Lebesgue function for polynomial interpolation. *SIAM J. Numer. Anal.*, 15(4):694–704, Aug. 1978.

[6] L. Brutman. Lebesgue functions for polynomial interpolation—a survey. *Annals Numer. Math.*, 4:111–127, 1997.

[7] A. Eisinberg, G. Fedele, and G. Franzè. Lebesgue constant for Lagrange interpolation on equidistant nodes. *Anal. Theory Appl.*, 20(4):323–331, Dec. 2004.

[8] J. F. Epperson. On the Runge example. *Am. Math. Mon.*, 94(4):329–341, Apr. 1987.

[9] M. S. Floater and K. Hormann. Barycentric rational interpolation with no poles and high rates of approximation. *Numer. Math.*, 107(2):315–331, Aug. 2007.

[10] P. Henrici. *Essentials of Numerical Analysis with Pocket Calculator Demonstrations.* Wiley, New York, 1982.

[11] G. Klein and J.-P. Berrut. Linear barycentric rational quadrature. *BIT*, 2011. `http://dx.doi.org/10.1007/s10543-011-0357-x`.

[12] G. Klein and J.-P. Berrut. Linear rational finite differences from derivatives of barycentric rational interpolants. *SIAM J. Numer. Anal.*, 2012. accepted for publication.

[13] R. B. Platte, L. N. Trefethen, and A. B. J. Kuijlaars. Impossibility of fast stable approximation of analytic functions from equispaced samples. *SIAM Rev.*, 53(2):308–318, June 2011.

[14] M. J. D. Powell. *Approximation Theory and Methods.* Cambridge University Press, Cambridge, 1981.

[15] T. J. Rivlin. The Lebesgue constants for polynomial interpolation. In H. G. Garnir, K. R. Unni, and J. H. Williamson, editors, *Functional Analysis and its Applications*, volume 399 of *Lecture Notes in Mathematics*, pages 422–437. Springer, Berlin, 1974.

[16] C. Runge. Über empirische Funktionen und die Interpolation zwischen äquidistanten Ordinaten. *Zeit. Math. Phys.*, 46:224–243, 1901.

[17] A. Schönhage. Fehlerfortpflanzung bei Interpolation. *Numer. Math.*, 3(1):62–71, 1961.

[18] S. J. Smith. Lebesgue constants in polynomial interpolation. *Annales Math. Inf.*, 33:109–123, 2006.

[19] L. N. Trefethen and J. A. C. Weideman. Two results on polynomial interpolation in equally spaced points. *J. Approx. Theory*, 65(3):247–260, June 1991.