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A FIRST ATTEMPT TO FRACTAL MOSAICS

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ABSTRACT

A new model for random tessellations having a fractal component is introduced. An explicit formula for the Hausdorff dimension is given and the exact gauge function of its Hausdorff measure is calculated. Moreover, fractal curvatures and mean fractal curvatures are considered. A theoretical result about the relation of these quantities is shown and is demonstrated numerically by an example.

Keywords: Random Fractal, Random Tessellation, Hausdorff Dimension, Curvatures, Mean Curvatures, Fractal Curvatures.

INTRODUCTION

Structures arising in biology, geology or medical sciences are often modeled as random tessellations (also called mosaics), see Stoyan *et al.* (1995). To obtain more complicated models which are better adapted to the observed situations, one uses operations on the space of random tessellations such as superposition, iteration (or nesting) or aggregation. It is a disadvantage of the known models and operations that they can't handle a fine (or micro-) structure such as hair line cracks for example. To include such phenomena it seems to be necessary to give up some properties of random mosaics and to allow them (at least in the limit of some limit procedure) to be *fractal* in some sense. Thereby we want the mosaic itself to be a fractal set and not only the cell boundaries (the last feature can be obtained by the operation of aggregation). It was therefore the aim of the authors diploma thesis Thaele (2007) to study a first model having the features described above. The motivation for our approach came thereby from the papers Nagel *et al.* (2003) and Nagel *et al.* (2005), which study the iteration of stationary random tessellations in the classical sense. An appropriate randomization of this operation leads to the notion of random iteration of tessellations, which will be explained below. The second main task was the choice of a class of tessellations for which direction computations can be made. Expecting a random fractal in the limit we decided to restrict our model to the self-similar case, because self-similarity is a well established concept in fractal geometry and explicit formulas are available in this case. Much more work has to be done to extend the results to some classes of random tessellations. It seems also to be ambiguous that explicit formulas can be obtained in such more general cases.

Another aim is – beside the calculation of the Hausdorff dimension and moreover an exact gauge function for the Hausdorff measure – the study of second order quantities for the limits sets of randomly iterated self-similar tessellations. The notion of fractal curvatures and fractal curvatures measures was already introduced in Winter (2008) for deterministic self-similar fractals satisfying the open-set condition. In the random case there are two possible generalizations of this concept: 1. the fractal curvatures of each realization, 2. mean fractal curvatures. Moreover one has to ensure that these quantities are well defined in our random setting. This is the most technical part of the work. As a main result we obtain that all quantities are indeed well defined and that the mean fractal curvatures can be calculated from only one random sample. This result comes from a comparison between the standard renewal theorem (in its L_1-version) and the renewal theorem for branching random walks. The numerical results for our special random case, which will be presented below, seem to be even more accurate than in the deterministic case.

The The model and the results on Hausdorff dimension were obtained in the authors Diploma thesis Thaele (2007). However, the results on fractal curvatures are new and their proofs will be shown elsewhere.

THE MODEL

A polyhedron P in $\mathbb{R}^d$ (which is not necessarily convex) is called *self-similar*, if there exists a natural number $n \in \mathbb{N}$ and polyhedra $P_1, \ldots, P_n \subset \mathbb{R}^d$ similar to P, intersecting each other only in their boundaries, such that

$$P = \bigcup_{k=1}^{n} P_k.$$

If in addition all P_k are congruent, we call P *replicating*. Examples for replicating polyhedra are the cubes $[0,1]^d$ or the standard simplices Δ^d. A (proper) self-similar dissection of a square for example is shown in Fig. 1. For a discussion of the more complicated higher dimensional case see Hertel (2005) and Osburg (2004) and the references therein.

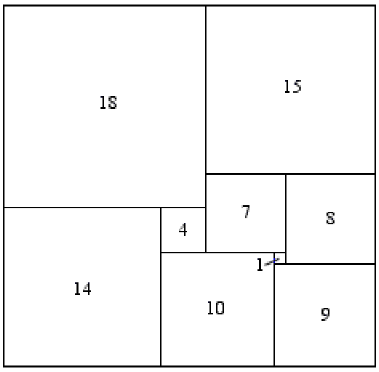

Fig. 1: Self-similar dissection of a square

By a *tessellation* or a *mosaic* M of $\mathbb{R}^d$ we understand a contable family of polyhedra $P_1, P_2, \ldots$ (often called *cells*) with following properties:

–$\bigcup_{k=1}^{\infty} P_k = \mathbb{R}^d$,

–a bounded set in $\mathbb{R}^d$ intersects only a finite number of polyhedra,

–the interiors of two different polyhedra are disjoint.

A tessellation is called *self-similar*, if all polyhedra are congruent (this assumption can be omitted, but is included for holding formulas simpler) and each polyhedron P_k is self-similar and *replicating*, if each P_k admits a replicating dissection as described above. Examples for tessellations in general can be found in Stoyan *et al.* (1995). An example for a self-similar tessellation is given by a tessellation, whose cells P_k are congruent (squares or higher dimensional) cubes. More complicated examples were constructed in Hertel (2005).

We describe now the operation of random iteration of tessellations. This is a randomized version of the definition used in Nagel *et al.* (2003) or Nagel *et al.* (2005), but restricted to the case of self-similar tessellations. Let M be a self-similar tessellation with a fixed self-similar pattern. With each cell P_{k_1} ($k_1 \in \mathbb{N}$) of M we associate now a random variable X_{k_1} with

$$\mathbb{P}(X_{k_1} = 1) = 1 - \mathbb{P}(X_{k_1} = 0) = p,$$

where $p \in [0,1]$ is a fixed model parameter (we will here assume maximal independence assumptions, i.e. we assume all random variables which occur within the paper to be independent). A random tessellation $I_1(M,p)$ is now obtained from the deterministic mosaic M by dividing all cells P_{k_1} with $X_{k_1} = 1$ according to their fixed self-similar dissection (if there is more than one possibility we choose one of them). The cells of $I_1(M,p)$ are denoted by $P_{k_1 k_2}$ and we associate with each $P_{k_1 k_2}$ the random variable $X_{k_1 k_2}$ with the same distribution as the X_{k_1} from above. The tessellation $I_2(M,p)$ is now obtained from $I_1(M,p)$ by dividing all cells $P_{k_1 k_2}$ according to their fixed self-similar dissection iff $X_{k_1} \cdot X_{k_1 k_2} = 1$, i.e. none of the factors equals zero. Proceeding in this way we obtain the *n-fold random iteration* $I_{n+1}(M,p)$ from $I_n(M,p)$ by dividing all cells $P_{k_1 \ldots k_n}$ iff $X_{k_1} \cdots X_{k_n} \neq 0$. For $n = 1,2,3,4$ this method is illustrated in Fig. 2.

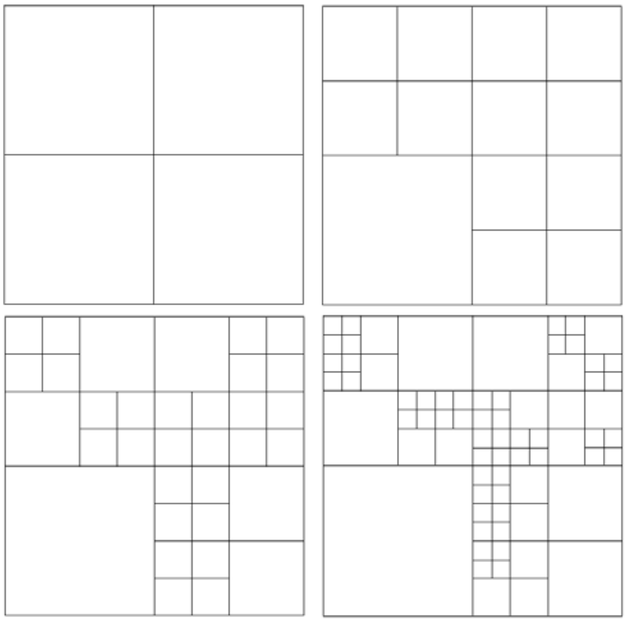

Fig. 2: The random iterations $I_n(M,p)$, $n = 1,2,3,4$ for a square tessellation with $p = 1/2$

We are here not interested in the tessellations $I_n(M,p)$, but in the limit as n tends to infinity. We therefore define the *limit set*

$$I_\infty(M,p) := \overline{\bigcup_{n=1}^{\infty} \{\partial P_{k_1 \ldots k_n} : P_{k_1 \ldots k_n} \in I_n(M,p)\}},$$

where ∂P is the boundary of the cell P and $\overline{A}$ denotes the topological closure of a set A. $I_\infty(M,p)$ is the closure of the union of all cell boundaries of cells, which were constructed in some random iteration step of the tessellation M with parameter p.

We like to motivate our definition a bit more: We wish to construct a tessellation which contains some fractal component. All cells involved have Hausdorff dimension d and so their union. It is therefore reasonable to consider the union of boundaries, which have Hausdorff dimension $d-1$. Since Hausdorff dimension is σ-stable (see the next section), the set

$$\bigcup_{n=1}^{\infty} \{\partial P_{k_1 \dots k_n} : P_{k_1 \dots k_n} \in I_n(M,p)\}$$

has Hausdorff dimension $d-1$, too. We therefore took the closure, hoping that $I_\infty(M,p)$ has Hausdorff dimension $> d-1$ (at least for some parameters p). We like to point out that similar approaches are well known in fractal geometry, see Falconer (1990). We further like to point out, that $I_\infty(M,p)$ contains a random fractal component as well as cell boundaries of usual polyhedra (which of course do not contribute to fractal properties of this set).
We consider as an example a 9-replicating mosaic of $\mathbb{R}^2$ by squares. For $p=0.2$, $p=0.5$ and $p=0.8$, a simulation of a random limit set is shown in Figures 3,4 and 5.

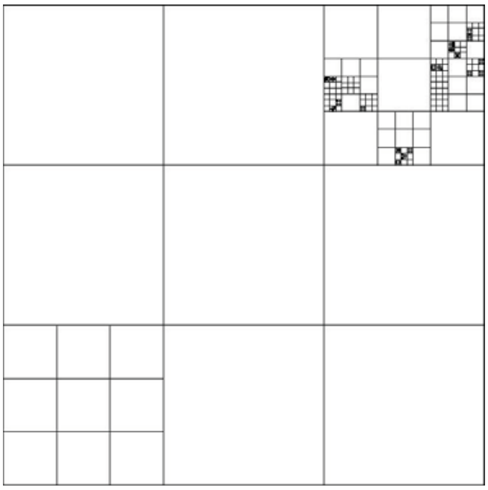

Fig. 3: A limit set for $p=0.2$

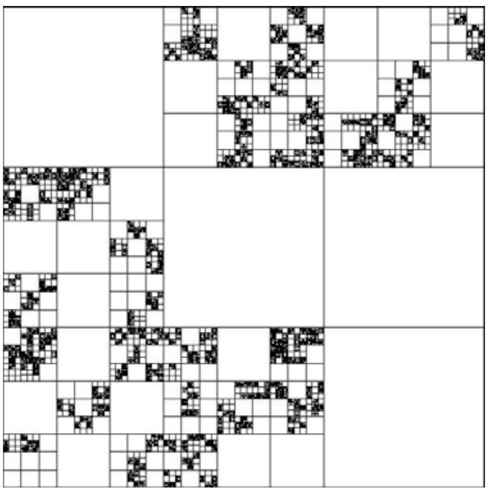

Fig. 4: A limit set for $p=0.5$

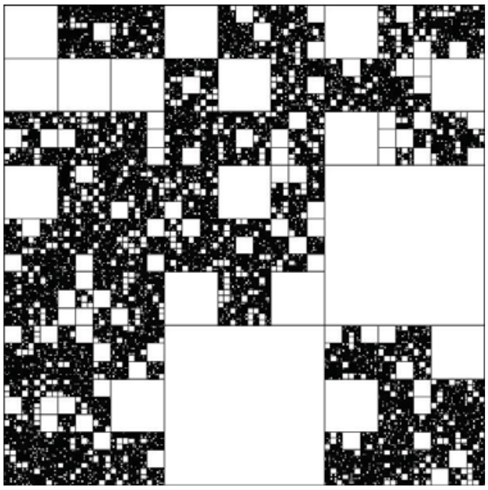

Fig. 5: A limit set for $p=0.8$

RESULTS

HAUSDORFF MEASURE AND HAUSDORFF DIMENSION

We start with recalling the basic definitions: Let $A \subset \mathbb{R}^d$ and $h : [0,\delta] \to [0,\infty)$, for some $\delta > 0$, be right continuous and non-decreasing with $h(t) > 0$ for $t > 0$ and $\lim_{t\to 0} h(t) = 0$. Assume furthermore that there exists a constant $K > 0$, such that $h(2t) \le Kh(t)$. The *Hausdorff measure* $\mathscr{H}^h(A)$ of $A \subset \mathbb{R}^d$ wrt. the gauge function h is defined by

$$\mathscr{H}^h(A) := \lim_{\delta\to 0} \inf \left\{ \sum_{k=1}^{\infty} h(|A_k|) : A \subset \bigcup_{k=1}^{\infty} A_k, |A_k| < \delta \right\},$$

where the infimum is taken over all coverings $(A_k)_{k=1}^{\infty}$ of A. In the definition, $|A_k|$ is the diameter of the set A_k. The measure $\mathscr{H}^h$ is a Borel regular Borel measure. In the case $h(t) = t^s$ we will write $\mathscr{H}^s$ instead of $\mathscr{H}^{t^s}$. We call a gauge function *exact* for a set A, if $0 < \mathscr{H}^h(A) < \infty$. The *Hausdorff dimension* $dim_H\ A$ of a set $A \subset \mathbb{R}^d$ is the unique $D \ge 0$ with the property that

$$\mathscr{H}^s(A) = \begin{cases} +\infty & : s < D \\ 0 & : s > D. \end{cases}$$

The Hausdorff dimension is monotone and σ-stable, which means that

$$dim_H \bigcup_{k=1}^{\infty} A_k = \sup_k\ dim_H\ A_k.$$

It is important to note that there exist sets $A \subset \mathbb{R}^d$ (for example our sets $I_\infty(M,p)$ or the graph or the zero set of the Brownian motion), having Hausdorff dimension D and for which $\mathscr{H}^D(A) = 0$ or $= +\infty$ holds. But there may exist a function h (in the case

of Brownian motion for example the function $h(t) = \sqrt{2t \ln\ln t^{-1}}$, for our sets see Theorem 3.1 below), for which $\mathscr{H}^h(A) \in (0,\infty)$. This shows that an exact gauge function contains more information about a set than Hausdorff dimension alone. It is therefore desirable to calculate for our limit sets $I_\infty(M,p)$ both, its Hausdorff dimension as well as an exact gauge function.
For the formulation of our first main result we need to introduce a few notation. With a self-similar dissection $P = P_1 \cup \ldots \cup P_n$ of a polyhedron P there are associated n contraction ratios $r_1,\ldots,r_n$, which are given by the relation $r_k P = P_k$, $k = 1,\ldots,n$. In the case of a replicating dissection these contraction ratios are necessarily equal and we will write $r = r_1 = \ldots = r_n$ in this special case. The first main result of Thaele (2007) reads now as follows:

Theorem 3.1 *Let $p \in [0,1]$ and M be a self-similar tessellation of $\mathbb{R}^d$ with associated contraction ratios $r_1,\ldots,r_n$ and assume that for all $k = 1,\ldots,n$ we have $0 < r_k < 1$. Then $I_\infty(M,p)$ has Hausdorff dimension $D = \max\{d-1,\alpha\}$ and α obeys*

$$\sum_{k=1}^{n} r_k^\alpha = \frac{1}{p}.$$

Moreover

$$h(t) = t^\alpha \left(\ln\ln\frac{1}{t}\right)^{1-\frac{\alpha}{d}}$$

is a exact gauge function for $I_\infty(M,p)$ if $(\sum_{k=1}^n r_i)^{-1} \leq p$ and $h(t) = t^{d-1}$ is an exact gauge function in the case $(\sum_{k=1}^n r_i)^{-1} > p$.

In the special case of a replicating tessellation M of $\mathbb{R}^d$ we can give a more explicit formula for the Hausdorff dimension:

$$D = dim_H\, I_\infty(M,p) = \max\left\{d-1, d-\frac{\ln p}{\ln r}\right\}.$$

For the exact gauge function we obtain

$$h(t) = \begin{cases} t^D \left(\ln\ln\frac{1}{t}\right)^{1-\frac{D}{d}} & : p \geq r \\ t^{d-1} & : p < r. \end{cases}$$

We consider again our example from above and the Hausdorff dimension of its limit set as a function of p. A plot of this function is shown in Fig. 6.

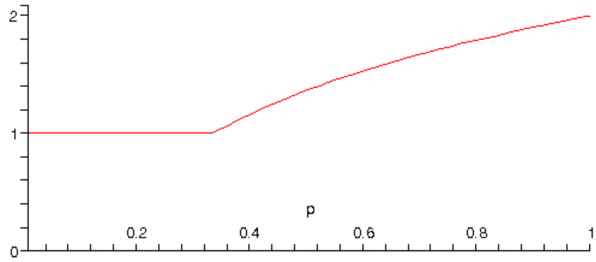

Fig. 6: Hausdorff dimension of the limit set of the example as a function of $p \in [0,1]$

To get an impression of the gauge function $h(t)$ (red) we compare it in the next plot (Fig. 7) with the function t^D (green).

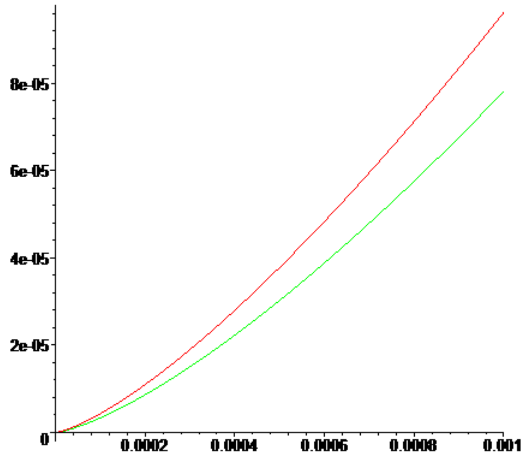

Fig. 7: Exact gauge function for $p = 0.8$

At the end of this section we want to give an idea of the proof of Theorem 3.1. The main step is to identify the limit set $I_\infty(M,p)$ with the image (under a suitable projection map) of the boundary of a certain random tree. This random set is constructed by a standard Galton-Watson process in the replicating case and by a Crump-Mode-Jagers (CMJ) process in the general case. This identification allows us to apply known results about the Hausdorff dimension and the exact gauge function of the boundary of such a random tree as for example Liu (1996) and Hambly *et al.* (2003). The retranslation into the geometric setting and an application of the so-called mass-distribution principle (see e.g. Falconer (1990)) gives now the desired result. The full details can be found in Thaele (2007).

FRACTAL CURVATURES AND MEAN CURVATURES

It is easy to construct two geometrically complete different limit sets of self-similar mosaics, whose

Hausdorff dimension and even whose exact gauge functions agree. One is therefore looking for other geometric quantities, which reflect finer properties of these sets (and which may help to distinguish between different limit sets having the same Hausdorff dimension). A first attempt in this direction was made by Winter (2008), where curvatures and curvature measures for a class of deterministic self-similar fractals were introduced. The paper contains also several examples of fractals with equal Hausdorff dimension, which can be distinguished by using the numerical values of their fractal curvatures. It is therefore the aim of this section to introduce fractal curvatures and mean fractal curvatures for the limit sets $I_\infty(M,p)$.

We start with recalling the definition of curvatures for polyconvex sets (i.e. finite unions of convex sets). Let $A \subset \mathbb{R}^d$ be polyconvex and $r > 0$. Then the volume of the r-parallel set $A_r := \{x \in \mathbb{R}^d : dist(x,A) \leq r\}$ can be expressed as a polynomial of degree d:

$$vol(A_r) = \sum_{k=0}^{d} \omega_{d-k} C_k(A) r^{d-k},$$

where ω_k is the volume of the k-dimensional unit ball. The coefficients C_k are called *curvatures* (or intrinsic volumes) of the set A. In particular $C_d(A)$ is the volume of A, $2C_{d-1}(A)$ is ($(d-1)$-dimensional) surface area of the boundary of A and $C_0(A)$ is the Euler characteristic (in the sense of algebraic topology) of A.

The idea is now the following: We approximate our limit sets $I_\infty(M,p)$ by their r-parallel sets, consider for $k = 0,\dots,d$ the curvatures $C_k((I_\infty(M,p))_r)$ (provided that they are well defined) and take the limit as r tends to zero. Unfortunately these limits fail to exist in 'almost all' cases. We therefore introduce the rescaling factor r^{D-k} and consider the limit

$$C_k^f(I_\infty(M,p)) := \lim_{r\to\infty} r^{D-k} C_k((I_\infty(M,p))_r), \ \ k = 0,\dots,d,$$

where D is the Hausdorff dimension of $I_\infty(M,p)$ given by Theorem 3.1. But also this limit fails to exist in some cases, in particular in our square tiling example. It turns out, that the Cesáro average on a logarithmic scale has a much better convergence behavior. We therefore define the *fractal curvatures* $C_k^f(I_\infty(M,p))$ for $k = 0,\dots,d$ of the limit set $I_\infty(M,p)$ of a d-dimensional self-similar mosaic M by

$$C_k^f(I_\infty(M,p)) :=$$

$$\lim_{\delta\to 0} \frac{1}{-\ln\delta} \int_\delta^1 r^{D-k} C_k((I_\infty(M,p))_r) \frac{dr}{r}.$$

This approach leads in $\mathbb{R}^d$ to $d+1$ parameters, which reflect the geometry of the fractal set $I_\infty(M,p)$. Note, that there is no abuse of notation when we use the same notation for the limits of rescaled curvatures in the ordinary as well as Cesáro averaged sense, since they agree whenever the ordinary limit exists. It is therefore sufficient to work only with the Cesáro version.

Regarding the above definition of $C_k^f(I_\infty(M,p))$ there arise naturally two main questions:

1.Are the curvatures $C_k((I_\infty(M,p))_r)$ well defined for any $k = 0,\dots,d$ and any $r > 0$?

2.How can we show convergence of the limit?

The first question can easily be answered: Yes, they are well defined, since $(I_\infty(M,p))_r$ is for any $r > 0$ a polyconvex set. The second question is much harder. The main idea is the following: First replace r by e^{-t}. Then establish a relationship between $e^{(k-D)t} C_k((I_\infty(M,p))_{e^{-t}})$ and $e^{(k-D)s} C_k((I_\infty(M,p))_{e^{-s}})$ for $s < t$. This relationship turns out to be the renewal equation of a certain branching random walk. Stochastic processes of this type were considered by several author, see Nerman (1981), Gatzouras (2000a) (a first application to self-similar fractals was shown in Gatzouras (2000b)). We apply in a next step the renewal theorem for branching random walks, which ensures that the limit $C_k^f(I_\infty(M,p))$ exists and is positive and finite. It remains to show that the assumptions of the renewal theorem are fulfilled in our case, which is the hard part of the work. For the case of (general) self-similar random fractals this was done in Zähle (2009) under some additional assumptions, which are not necessary in our case.

Since the limit sets $I_\infty(M,p)$ are random, the fractal curvatures $C_k^f(I_\infty(M,p))$ are random, too. But the renewal theorem shows even more than only the convergence. We get, that $C_k^f(I_\infty(M,p))$ is a random multiple, which only depends on the distribution of $I_\infty(M,p)$ but not on k, of a deterministic integral I. So we can write

$$C_k^f(I_\infty(M,p)(\omega)) = X(\omega) \cdot I(k,p,M).$$

This shows that the quotients

$$c_{k,l} := \frac{C_k^f(I_\infty(M,p))}{C_l^f(I_\infty(M,p))} = \frac{I(k,p,M)}{I(l,p,M)}$$

are deterministic constants and we are now going to characterize them in terms of the *mean fractal curvatures* of $I_\infty(M,p)$. They are defined by the same idea as above. For $k = 0,\dots,d$, we put

$$\overline{C_k^f}(I_\infty(M,p)) :=$$

$$\lim_{\delta\to 0} \frac{1}{-\ln\delta} \int_\delta^1 r^{D-k} \mathbb{E} C_k((I_\infty(M,p))_r) \frac{dr}{r}.$$

By similar ideas as in the case of the (random) fractal curvatures, it can be shown that $\overline{C_k^f}(I_\infty(M,p))$ exists and is positive and finite. But in the case of mean fractal curvatures the classical renewal theorem in its L_1-version is sufficient. Also in this case, the renewal theorem shows even more than the existence. We obtain

$$\overline{C_k^f}(I_\infty(M,p)) = I(k,p,M),$$

where $I(k,p,M)$ is the same deterministic integral as above. Thus, we get at the end

Theorem 3.2 *For the limit set $I_\infty(M,p)$ of a random self-similar mosaic M in $\mathbb{R}^d$, the fractal curvatures and mean fractal curvatures $C_k^f(I_\infty(M,p))$ and $\overline{C_k^f}(I_\infty(M,p))$ exist and are positive and finite. Moreover*

$$\frac{C_k^f(I_\infty(M,p))}{C_l^f(I_\infty(M,p))} = \frac{\overline{C_k^f}(I_\infty(M,p))}{\overline{C_l^f}(I_\infty(M,p))} = \frac{I(k,p,M)}{I(l,p,M)} := c_{k,l}$$

is a constant only depending on k, l, M and p.

This shows, that from only one measurement we can obtain a mean value, i.e. the mean value over the full sample.
To demonstrate this numerically, we come back to our square tiling example from above. We estimate the Hausdorff dimension and the fractal curvatures C_0^f (fractal Euler number), C_1^f (fractal boundary length) and C_2^f (Minkowski content) for the parameter $p = \frac{1}{2}$ of 6 different realizations of $I_\infty(M,p)$. The results are summarized in the following table:

Nr.	dim_H	C_0^f	C_1^f	C_2^f
1	1.5090	−1893.78	18340.17	64169.08
2	1.5519	−2396.06	21096.40	77923.86
3	1.4452	−1690.27	17737.18	57018.47
4	1.5069	−1898.76	18393.58	63596.43
5	1.4164	−1221.33	14481.39	45134.82
6	1.4505	−1498.31	16437.67	53719.87

From these values we can now compute the quotients $c_{k,l}$:

Nr.	$c_{0,1}$	$c_{0,2}$	$c_{1,2}$
1	−0.095295	−0.029644	0.311077
2	−0.113576	−0.030748	0.270730
3	−0.103259	−0.029512	0.285810
4	−0.103229	−0.029856	0.289223
5	−0.084338	−0.027059	0.320847
6	−0.091151	−0.027891	0.305988

The values confirm experimentally the theoretical results from Theorem 3.2 and show that the numerical methods – which were originally developed for the deterministic case, see Straka *et al.* (2009) – seem to be stable with respect to random perturbations. Moreover they seem to work even more accurate in the random case, which was already expected. More details, discussions and examples will appear in Straka *et al.* (2009).

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