

Diffusion approach for community discovering within the complex networks: LiveJournal study

Pavel Zakharov

Department of Physics, University of Fribourg, CH-1700, Switzerland

Abstract

The diffusion approach of concentration mapping is used to discover communities in the directional friendship network of LiveJournal users. We show that this Internet-based social network has a power-law region in degree distribution with exponent $\gamma = 3.45$. It is also a small-world network with high clustering of nodes. To study the community structure we simulate diffusion of a virtual substance immersed in such a network as in a multi-dimensional porous system. By analyzing concentration profiles at intermediate stage of the diffusion process the well-interconnected cliques of users can be identified as nodes with equal values of concentration.

Keywords: Social networks; Scale-free networks; Community structure; Diffusion

1. Introduction

In recent years there has been an enormous break-through in research of complex networks due to the application of statistical physics methodology [1–3]. Many different complex systems instead of being completely random prove to have signatures of organization such as clustering and power-law distribution of links. Together with the small-world property [4] these are the inherent features of an extremely wide variety of systems such as the World-Wide Web [5–7], Internet [8], collaboration networks of movie actors [9,10] and scientists [10], the web of human sexual contacts [11] and many others. In spite of the fact that some concepts of complex networks theory were originally introduced in sociology the statistical study of social networks is complicated by the difficulty in reliable data collection due to certain privacy and ethical reasons. One of the solutions for this problem is the analysis of collaboration networks [9,10], e-mail interactions [12–14], instant messaging [15] and online blogging [16–19]. Here we studied basic structural properties of LiveJournal (LJ) blog service social network and demonstrated the diffusion-motivated method to discover communities on the case of this network.

E-mail address: Pavel.Zakharov@unifr.ch.

2. LiveJournal network

LJ is an online web-based journal service with an emphasis on users interactions [20]. In January 2006 it had 9.3×10^6 users in total, 2.0×10^6 of them were *active in some way* according to official LJ statistics [21]. The essential feature of LJ service is the “friends” concept which helps users to organize their reading preferences and provides security regulations for their journal entries and personal data. Friends list is an open information and can be accessed through a conventional WWW interface or through a dedicated bot interface provided by LJ system.

Data collection was performed by crawler programs running simultaneously on two computers and exploring the LJ space by following directional friendship links starting from two users with a large number of incoming friendship links. For each user the crawler was obtaining his friends list (outgoing links) and the number of users who have the given user in their friends list (incoming links). Each user from the friends list which was not yet explored by the crawler was added to the end of the processing queue if he was not already there. If the user was in the queue his queue score was incremented every time he was found in someones’ friends list. Users with higher queue scores were processed first. This ensured fast collection of the essential part of the network. Basically this algorithm is a modification of Tarjan’s depth-first search algorithm for finding the connected component of a graph [22,23]. Total time of collection was 14 days with the total number of discovered users 3746264 found in a connected component which is composed of strongly connected component and out-component [24]. We are aware that during the time of collection the network was undergoing continuous changes. We estimated the number of users deleted from the LJ database but still present in the friends lists was less than 0.1% which makes us believe that the evolution of LJ network did not influence our statistics much.

The estimated probability distribution functions of in- and out-degree are presented in log-log scale in Fig. 1. The estimated mean of the numbers of friendship links is $\langle k_{out} \rangle = \langle k_{in} \rangle = 15.91$. The average in-to-out ratio $\langle k_{in}/k_{out} \rangle = 1.157$.

There are several technical restrictions for the degrees: maximum number of friends per user is 750 and only 150 of them can be listed on the users’ info page and can be effortlessly accessed by the LJ users. From our experience LJ bots interface does have some problems listing the users who consider a certain user as a friend if there are more than 2500 of them hence we cut the data at $k_{max}^in = 2500$.

As one can see from the Fig. 1 in- and out-degree distributions reveal a power-law decay $P(k) \sim k^{-\gamma}$ for $k > 100$ with the value of the exponent $\gamma_{in} \approx \gamma_{out} = 3.45 \pm 0.05$ which is surprisingly close to the values $\gamma_{in} \approx \gamma_{out} \approx 3.4$ obtained by Liljeros et al. for sexual contacts [11]. Scaling of the distributions contradicts the results of Liben-Nowell et al. [19] who reported parabolic shape of LJ degrees distributions. The skewness of the distributions in our case can be explained by the social origin of LJ network. As it is pointed out by Jin et al.

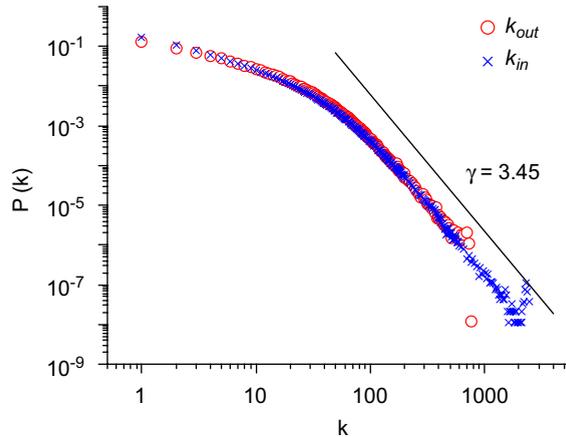


Fig. 1. Probability density functions of in- and out-degrees for LiveJournal users. The line shows a slope of -3.45 which equally well fits $P(k_{in})$ and $P(k_{out})$.

[25] degree distribution for social networks does not appear to follow power-law distribution due to the cost in terms of time and efforts to support friendship. In the case of LJ network the cost of friendship is the size of friends feed which accumulates all the recent entries of the user's friends. We can also separate two classes of LJ users: "readers" and "writers". The first are mainly using their accounts to read the journals of others. They update journals only episodically and are not deeply involved in LJ community life. They do not have many incoming and outgoing links and they are responsible for skewness of the distributions for $k < 100$. Meanwhile active "writers", representing minority of the registered users exploit full capacity of LJ system. They spend much time participating in LJ community life, and they have a larger number of incoming and outgoing links which are distributed by power-law.

The origin of power-law region in the distributions can be explained by continuous evaluation and self-organization of the LJ network and preferential attachment mechanism similar to the general WWW growth mechanism [26]. Once an interesting journal gets popular it will be cited and promoted in the journals of its readers which will help to further increase its popularity which leads to a "rich-get-richer" effect occurring in many network systems [2,26]. However linear growth with linear preferential attachment protocol leads to a power-law degree distribution with $\gamma = 3$ which is smaller than the exponent obtained for our study. Larger values of exponent can be explained by alternative growth mechanisms: preferential attachment with rewiring [27] and copying mechanism [6,7]. Rewiring in LJ system implies that users are not only establishing new friendship links but also breaking the old ones while copying occurs when the user inherits part of the friendship connections of his friends. The latter effect is called "transitivity" in sociology [28] and is responsible for users cliques formation or clustering.

We characterize clustering of LJ users by calculating the clustering coefficient as introduced by Watts and Strogatz [1,9]. It can be defined as the number of links between user's friends divided by the maximum possible number of links between them averaged over all users in the network. Consider the user i with k_{out}^i friends and with existing E_i links between them. Meanwhile the maximum possible number of directed links is $k_{out}^i(k_{out}^i - 1)$ and thus the clustering coefficient for the group of user i friends in the case of directed network can be defined as

$$C_i = \frac{E_i}{k_{out}^i(k_{out}^i - 1)}. \quad (1)$$

The average clustering coefficient for the whole network as calculated from our data is: $C = \langle C_i \rangle_{i=1..N} \approx 0.3302$. It is worth to compare this value to the clustering coefficient of a random directional Erdős-Rényi graph which can be found as $C_{rand} = \langle k \rangle / (N - 1)$ which for LJ network is ca. 4.24×10^{-6} . The fact that actual clustering coefficient for LJ network is nearly five orders of magnitude larger than it would be expected from randomly linked network with the same degree and size is a clear indication of high user clustering.

The peculiar feature of the LJ network is the high reciprocity [28] of friendship links. We found that 79.26% of links are bidirectional which means that this percentage of outgoing links is returned as incoming and *vice versa* the same percentage of incoming links originates from users friends. This value is higher than reciprocity 57% found for the WWW [14] which is the technical environment of LJ. Increasing reciprocity may be explained by social origin of LJ network. Due to the rules of social interactions user A usually feels obliged to establish a friendship connection to the user B if such a connection was already established by B to A . Another explanation for high reciprocity is that often relations in LJ space are based on real-life people relations which means that LJ users are linking to the other users which are their friends in the real world. In this case the LJ network directly inherits the unidirectional structure of the underlying social network.

In order to characterize small-world properties of LJ network we estimated the probability distribution function $P_\ell(\ell)$ of the minimum path distance or hopcount between the nodes through directional links. The results are presented in Fig. 2. The average distance estimated for our set of data is $\langle \ell \rangle = 5.86$. Based on the recently obtained expression for the mean distance between the nodes in random networks by Fronczak et al. [29] the value of $\langle \ell \rangle$ can be estimated in the following way:

$$\langle \ell \rangle_{th} = \frac{-2\langle \ln h \rangle + \ln N + \ln \langle h^2 \rangle + \gamma_e}{\ln N + \ln \langle h^2 \rangle - \ln \beta} + \frac{1}{2}, \quad (2)$$

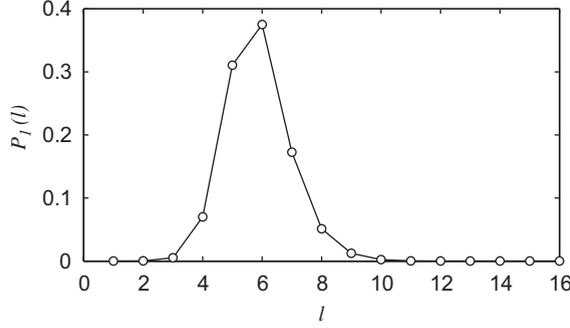


Fig. 2. Probability distribution function of the minimum path length between LiveJournal users through the directional friends links.

where N is the size of the network, $\langle h \rangle = \langle k_{out} \rangle$, $\langle h^2 \rangle = \langle k_{out}(k_{out} - 1) \rangle$, $\gamma_e \approx 0.577$ is the Euler–Mascheroni constant, and β is the constant which is equal to $\beta = \langle h \rangle N$ when the absence of the two-point correlations is assumed.

For the LJ data discussed here Eq. (2) gives the estimation of the mean distance: $\langle \ell \rangle_{th} = 4.746$ which is somewhat smaller than statistically obtained value. This might be due to the presence of the pairwise correlations of out-degrees and macroscopic structuring of the network which is discussed further.

3. Community discovering method

It seems to be quite natural for the nodes of the complex networks to aggregate into macroscopic structures with high internal links density and weak connection to the rest of the network. Such groups are often referred to as communities. Particular reasons for communities formation may depend on the type of the network but this feature proved to be quite universal and can be found in social, biological and computer networks [30,31]. Finding these structures within the network is the major step towards understanding its topology.

The problem of graph partitioning is a NP-complete problem which makes it almost inapplicable for large networks.

Recent advances in the study of complex networks stimulated the search of alternative techniques for community discovering and many original solutions were proposed [30–37]. These algorithms can be divided into two main classes: *divisible*, which hierarchically split the network by removing edges with the highest betweenness [30,33], and *agglomerative*, which start from the maximal community division when each node belongs to its own separate community and continuously merges these communities based on some parameter of nodes similarity [35,36] or optimizing the partitioning. In their recent work Clauset et al. [34] used the greedy optimization in order to maximize the *modularity* measure of partitioning quality [31,33]. Currently this method is one of the fastest and runs in time $O(MH \ln N)$, where $M = \langle k \rangle N$ is the number of edges in the network and H is the number of decomposition levels which is usually small ($H = O(\ln N)$) [34,35]. In a sparse network the degree is limited and $M = O(N)$ and so the complexity is $O(N \ln^2 N)$ which makes it fastest nowadays. Our approach is based on the simulation of a mass diffusion process in the complex network as in a multi-dimensional porous system with directional links following physical laws. The diffusion process initiated at one of the nodes by addition of the virtual ink produces a non-uniform mass distribution at the intermediate state which can be used to reveal well-interconnected communities within the complex network by selecting the nodes with similar concentration values. In this sense our method falls in the class of agglomerative techniques with the concentration as the similarity measure. However, it can be shown that the quantity $r_{AB} = |\ln \phi_A - \ln \phi_B|$, where ϕ_A and ϕ_B are two values of concentration in the nodes A and B , as the measure of distance between these nodes. Thus edge betweenness, characterized as the drop of the logarithm of concentration along the edge, can be used for hierarchical decomposition of the network.

The similar measure of distance between nodes based on the random walk has been recently introduced by Pons and Latapy [35] for the class of undirected networks. It is defined as the difference in probabilities for

a random walker to reach nodes A and B in certain number of steps t starting from some node Z . As these probabilities for a large t are mainly determined by the in-degrees of the nodes the values of distance should be normalized. A short number of steps t may depend on a particular network and should be known in advance. Pons and Latapy also pointed out conceptual difficulties of the random walk scheme application for the directed networks [35]. Several other diffusion motivated approaches proposed recently (e.g. [36–39]) are more or less consistent with random-walk analogy.

In our model we break the similarity with classical random walks and the theory of flows in the graph [40] in favor of a realistic physical picture. First, we allow nodes to accumulate substance by assigning to them infinite maximum capacity. The direct flow from the node A to the node B is possible if there is a directed link from A to B and $\phi_A > \phi_B$. The flow rate in this case depends on the concentration difference $\phi_B - \phi_A > 0$ and the out-degree k_{out} of the node A . In the case of $A < B$ no mass is delivered directly from A to B . Such rules in the limit of infinite time lead to equilibrium state with equal mass distribution which meets the physical expectations. However, classical random-walk approach will lead to gathering of random walkers in the out-component of the network.

Network links in our realization represent pipes (Fig. 3), directed links act as pipes allowing mass to pass in one direction. Mass propagation within the network system is driven by Fick's law of diffusion:

$$dM = -D \frac{\delta\phi}{\delta x} dS dt, \quad (3)$$

where dM is mass change, $\delta\phi/\delta x$ is concentration gradient and dS is an area element.

For our discrete system this implies that the rate of mass exchange between the neighboring nodes is proportional to the difference of masses in these nodes. Every node uses its outgoing links to deliver mass to its neighbors with a smaller amount of ink. The amount of ink $\Delta_{out}M_i$ delivered by the node to its i th neighbor is

$$\Delta_{out}M_i = -\frac{\alpha}{k_{out}}(M_0 - M_i), \quad (4)$$

where $M_0 > M_i$ and α is the coefficient determining the transfer rate and is constant for all nodes. We analyze the mass M contained in the node instead of the concentration ϕ assuming that all nodes have the same geometrical volume. The total delivered mass for a node is the following:

$$\Delta_{out}M = \sum_{i=1}^{k_{out}} \Delta_{out}M_i = -\alpha \left(M_0 - \frac{1}{k_{out}} \sum_{i=1}^{k_{out}} M_i \right) = -\alpha(M_0 - \bar{M}), \quad (5)$$

where \bar{M} is the mean ink mass in the neighboring nodes with smaller masses. Mass transfer in the pipe happens instantaneously. Thus we can apply mass conservation law and increase mass in the neighboring

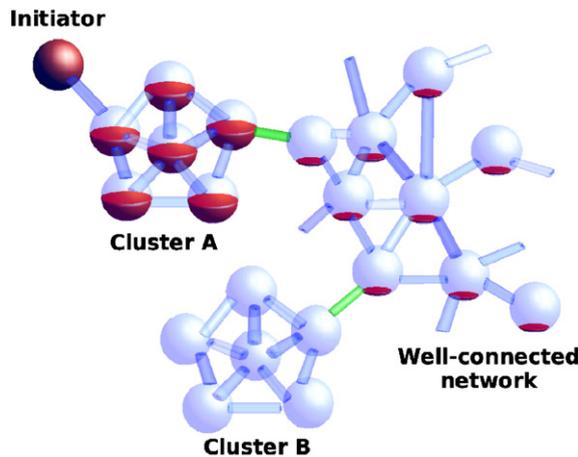


Fig. 3. Illustration of the community detection algorithm. After diffusion process starts from the initiator node virtual ink propagates through network links. Communities can be recognized as the groups of nodes with similar amount of ink.

nodes by the amount taken from the node:

$$\Delta_{out}M = - \sum_{i=1}^{k_{out}} \Delta_{in}M_i, \quad (6)$$

$$\Delta_{in}M = - \sum_{i=1}^{k_{in}} \Delta_{out}M_i. \quad (7)$$

The total change of mass at a certain node is composed of the loss of mass due to diffusion to the neighbors through outgoing links and gain of mass by the amount delivered from neighbors through incoming links: $\Delta M = \Delta_{in}M + \Delta_{out}M$. This conservation law is the extension of Kirchhoff's law [40] for the node with non-zero capacity.

In order to prevent inequality due to sequential nodes processing, mass changes for all nodes were calculated without actually changing the masses and then values of the masses in all nodes were updated. For the special case of absence of outgoing links $\Delta_{out}M = 0$ the specific node acts as a virtual ink absorber which can only gain ink from the neighbors but does not have ways to deliver it back. Nodes without incoming links are not considered due to their invisibility for the data collecting crawler and thus are absent in our database.

We start by putting an initial amount of ink of $M_0 = N$ mass units in one of the nodes which we call the *initiator*. Subsequently system is allowed to proceed to the equilibrium state by continuous mass redistribution within the network according to our rules. The expectation for an equilibrium state for a connected network system is equal distribution of mass M_0 among the nodes so that each of them ends up having $M_0/N = 1$ mass unit. While evolving to this state the system passes through non-equilibrium states with non-uniform mass distributions.

Imagine a cluster of well-connected nodes inside the network connected to the outside world only by few outgoing and incoming links. The ink diffusion inside the cluster is relatively fast due to the presence of a large number of exchange channels between the members and a high conductivity of the channels ensemble. Limited number of channels going outside the cluster forms the bottleneck for mass delivery. Under these conditions the flow rate between the members is much higher than between the members and non-members and dispersed ink will likely form an equi-concentrational volume within the cluster. Each cluster in this system with specific connection properties such as flow rate and distance from the initiator would have in each of its nodes the same concentration of ink with the value specific to the particular cluster. Thus by estimating the probability distribution function of concentration one can analyze non-uniformity of ink distribution and reveal separated clusters by determining the signatures of equi-concentration volumes.

The flow rate α from Eq. (4) can be selected from the half-interval (0;1] and defines the speed of simulation. Values larger than 0.5 are not desirable because they can cause concentration waves or back-reflections in some cases.

The proposed method does not aim to decompose the whole network on minimal clusters but to reveal significant clusters within the network. As we regard the network as an open system which does not have to be fully described by existing database we do not assign measure of clustering of the whole network like modularity proposed by Newman [32,33]. However, we can quantify the isolation of the individual community i by parameter of confinement K_i which is the characterization of assortative mixing of individual community. We can define K_i using notation of Newman [32] as following:

$$K_i = \frac{e_{ii}}{\sum_j e_{ij}} = \frac{e_{ii}}{b_i}, \quad (8)$$

where e_{ij} is the fraction of network edges connecting nodes of the community i to the community j and $\sum_j e_{ij} = b_i$ is the fraction of edges starting from the members of i . Thus parameter K_i defines the number of links connecting the nodes inside the community i as a fraction of the total number of links originating from the members of i .

4. Results and discussion

To test our method we performed ink diffusion simulations using our LJ database starting from different initiator nodes. Fig. 4 shows the relative mass decay as a function of simulation step number T for the flow rates $\alpha = 0.1, 0.25$ and 0.5 . User *doctor_livsy* with a high number of incoming links was chosen as the initiator node. As we will show later this user belongs to extremely confined Russian-speaking community. The inset of Fig. 4 shows the same data rescaled with respect to α . As one can see from the match of rescaled curves the dynamics of the process does not depend on the flow rate α in this range. The striking feature of the presented data is the obvious step-like form of the curves which is the effect of non-homogeneous structure of the LJ network. Flat parts of the $\Delta M/M$ curves correspond to the exponential decays of M which is the sign of non-restricted diffusion of ink. The first significant drop of the decay rate happens when $T\alpha \approx 5$ which is equal to the double radius of the community to which our initiator belongs. This corresponds to the moment when virtual ink fills the whole community and further expansion of filled area is impeded by the limited number of links going outside the community. So if it takes T_0 simulation steps for the virtual ink to reach the borders of the community it also takes T_0 simulation steps for the decay of concentration gradient to reach the initiator node and together this gives double size of the community. The second drop at $T\alpha \approx 22$ is not well pronounced and corresponds to the filling of the whole network. It is worth mentioning that such a stepwise dynamics is quite similar to the time dynamics of Kuramoto oscillators synchronization in complex networks reported recently by Arenas et al. [41] where synchronization patterns were used to uncover community structures in model networks.

As our community discovering algorithm is based on the detection of equi-concentration volumes we performed the calculation of the probability distribution function of M at two stages of virtual ink diffusion for $\alpha = 0.1$ (Fig. 5). One can see two well pronounced peaks on all plots which occurred to be the Russian-speaking community (larger values of mass M) and the rest of LJ network (broader peak at smaller values of M).

The dynamics of virtual ink distribution is presented in the Fig. 6. As it can be seen a distinct separation of the Russian community peak from the main peak is formed before step $T\alpha = 50$. At the latter stage it is quite stable and easily distinguishable up to iteration $T\alpha = 10^3$ which gives quite a long quasi-stationary stage that can be used for communities detection. It also demonstrates that the process of equi-concentrational volumes formation is much faster than the relaxation of the whole system.

If the initiator node is selected somewhere outside the community the splitting of the distribution peak is also observed but for this case average concentration within the Russian community is smaller compared to

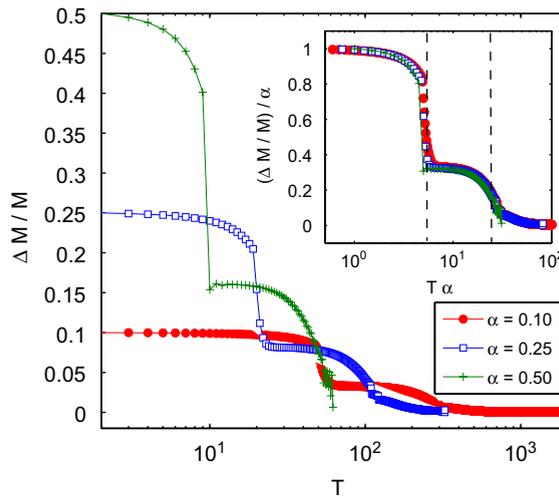


Fig. 4. Dynamics of relative concentration change in the initiator node *doctor_livsy* for different flow rates α . Inset shows rescaled data. Oscillatory parts were cut away.

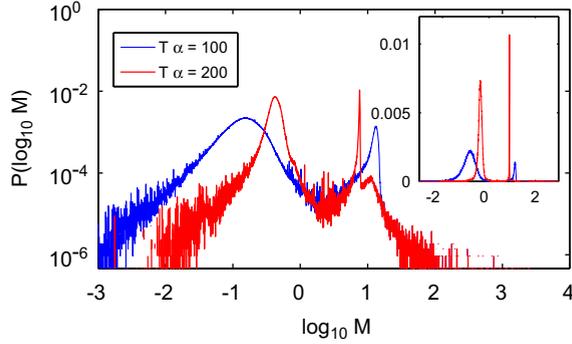


Fig. 5. Probability distribution functions of virtual ink concentration M at two stages of the diffusion process with $\alpha = 0.1$ and *doctor_livsy* as the initiator node. Inset represents the same data in linear scale. Two well pronounced peaks of two separated communities are clearly seen.

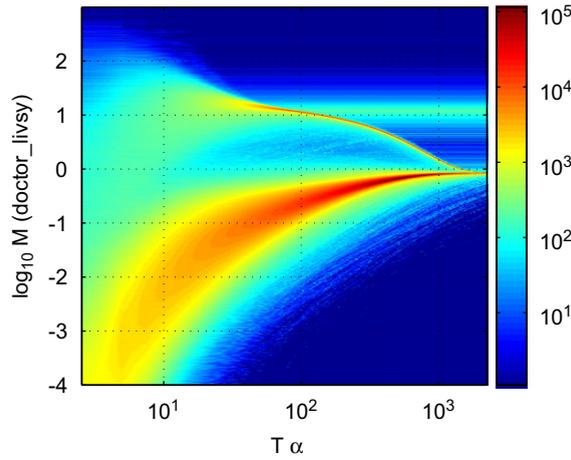


Fig. 6. Dynamics of virtual ink distribution within LJ network as a logarithmically color coded probability distribution function of the ink concentration (vertical axis) and simulation step (horizontal axis). Separation of Russian-speaking community (thin upper line, high concentration values) from general English-speaking (thicker lower line, lower concentration values) can be clearly seen.

the rest of the LJ nodes. This supports the expectations that if the community has a limited number of outgoing links it also lacks incoming links.

The accuracy of community discovering scheme can be improved by simultaneous simulation of the diffusion from two or more initiator nodes. Here we assigned two independent concentration values to a single node. All diffusion processes proceed without influencing each other. The LJ network can now be mapped as a probability distribution function of two concentrations and thus the community can be localized on a two-dimensional plot as shown in Fig. 7 for *doctor_livsy* and *future_visions* as the initiator nodes. One can see two main separated peaks corresponding to the major part of LJ network and the Russian-speaking community. The abundance of noise-like spots on the map corresponds to the small well-separated and well-linked communities existing in the network which are well localized.

The selection of nodes from a certain community can be performed by simple thresholding of the values of both concentrations. The group of nodes with the concentration values within the selected range which form the connected component in the network can be identified as the community. The ratio of the number of connected nodes to the total number of users with concentrations within the range defines the *specificity* of the method.

As the complete analysis of LJ community structure as well as the reasons of their formation is out of the scope of the current paper we will not list all user cliques found. However, in Table 1 we list the largest LJ

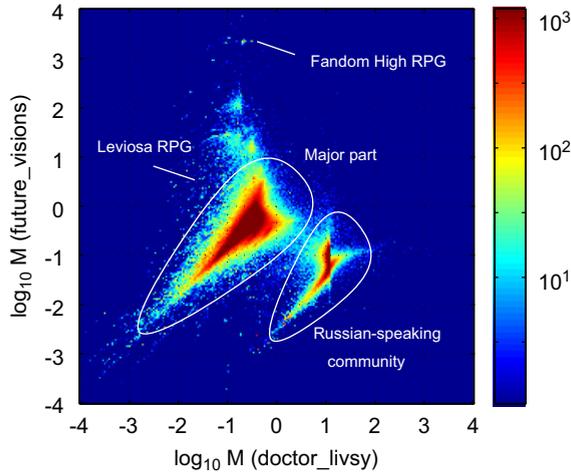


Fig. 7. Two-dimensional map of LJ users network obtained by concentration configurations of independent diffusion processes from two initiator nodes on the stage $T\alpha = 100$.

Table 1
Examples of discovered communities within LiveJournal userspace

Representing node	Number of users	Specificity (%)	Confinement K (%)	Comments
<i>doctor_livsy</i>	227314	99.89	98.34	Russian-speaking community ^a
<i>future_visions</i>	421	98.36	96.22	Fandom High Role-Playing Game community
<i>alected</i>	262	99.21	99.10	Leviosa Role-Playing Game community

^a92% of users have Cyrillic letters in their information pages or journals.

community and two smaller ones together with their parameters. The size of discovered Russian-speaking community is of the order of the total number of LJ users from the Russian Federation according to LJ database statistics [21] (232 241 users in January 2006). The obvious reason for the separation of this community with a very high value of confinement $K = 98.34\%$ is the prevailing usage of Russian language. We found by separate analysis of information pages and journal entries that 92% of the users within this community are using Cyrillic alphabet. The fact that the Russian LJ community differs from the rest of LJ network has been already pointed out by Internet observers (e.g., Ref. [42]). Curiously enough, even the Russian Internet hardware infrastructure (so-called autonomous systems) appeared to be relatively separated from the main layout of Internet as shown by Eriksen et al. [39].

The two other listed communities are the examples of surprisingly popular class of Role-Playing Game communities formed by the virtual users playing characters and writing their journals on behalf of these characters.

5. Conclusions

The LiveJournal friendship network was studied with the general approach developed for the complex networks and a power-law tail with exponent $\gamma = 3.45$ was found in the degree distributions. This network also demonstrated small-world property and high clustering.

To study the community structure we utilized the original diffusion approach. We found that diffusion in a network geometry with community structure leads to a peculiar phenomenon of formation of quasi-stationary equi-concentration volumes as shown by our simulation. This proves to be very useful for the detection of well-interconnected groups of nodes. With a limited number of parallel diffusion processes sufficient for a rough decomposition our method has an $O(N \ln N)$ complexity (each simulation step analyzes $M = \langle k \rangle N$

edges which for a sparse matrix $M = O(N)$ and the required number of steps is proportional to the diameter of the network which is $O(\ln N)$). It is currently one of the fastest algorithms and was applied for a huge directed network of LJ users containing several millions of nodes. To obtain results presented in this paper it takes only 1 or 2 h of desktop computer time. Moreover this method can be applied locally to a specific part of the network even with the lack of complete information about distant parts of the network. The sensitivity of decomposition can be tuned by increasing the number of initiator nodes with the limit of complete decomposition when every node acts like initiator of its own diffusion process.

Acknowledgments

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